Day 2

TU Delft

Wave equation

$$\nabla^2 p(\underline{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} p(\underline{r},t) = S(\underline{r},t)$$

↓ Fourier transform

$$\nabla^2 \hat{p}(\underline{r},\omega) + \frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega) = \hat{S}(\underline{r},\omega)$$

$$\nabla^2 \hat{G}(\underline{r},\omega) + \frac{\omega^2}{c^2}\hat{G}(\underline{r},\omega) = -\delta(\underline{r} - \underline{r}')$$

Christiaan Huygens

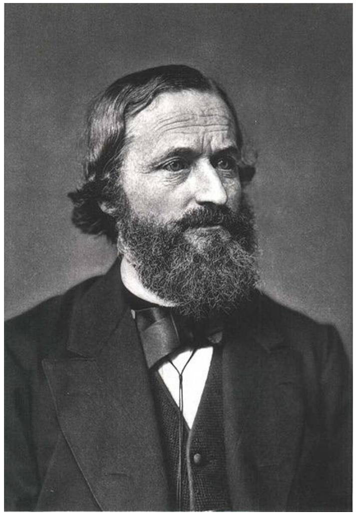
14 April 1629 (The Hague)
8 July 1695 (The Hague)

Augustin Jean Fresnel

10 May 1788 (Broglie)
14 July 1827 (Ville-d'Avray)

George Green

14 July 1793 (Sneinton)
31 May 1841 (Sneinton)



Gustav Robert Kirchhoff

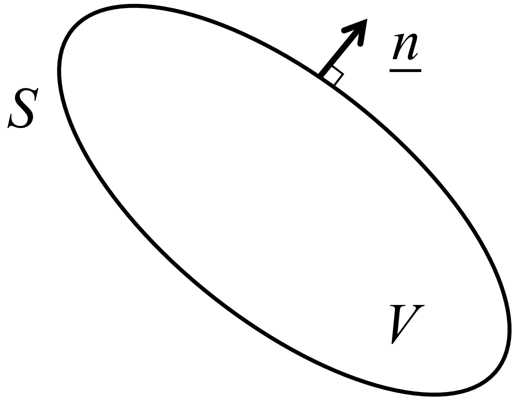
12 March 1824
(Koningsbergen)
17 Oct. 1887 (Berlin)

[1] http://nl.wikipedia.org/wiki/Christiaan_Huygens
[2] http://nl.wikipedia.org/wiki/Augustin_Jean_Fresnel
[3] http://www.nottingham.ac.uk/physics/about/history/george-green.aspx
[4] http://nl.wikipedia.org/wiki/Gustav_Robert_Kirchhoff

(Huygens – Fresnel) – Green – Kirchhoff – Rayleigh

- Wave-fields can be calculated as a function of space and time, from known values along a (closed) boundary.
- The oldest formulation of this process is Huygens' Principle. Later, Fresnel gave a more mathematical, but still somewhat heuristic description of wave-field extrapolation.
- Mathematically exact extrapolation of wave-fields is accomplished with the help of the Kirchhoff and Rayleigh integrals, which are based on Green's Theorem.
- The extrapolation algorithm is based on the wave equation and the causality of wave propagation.
- Wave-fields can be extrapolated forward and backward in time and space.

Green's Theorem



Consider Gauss' Theorem for any arbitrary vector field $\underline{a}(\underline{r})$ and arbitrary volume V, bounded by surface S :

$$\int_V \nabla \cdot \underline{a}(\underline{r})\, dV = \oint_S \underline{a}(\underline{r}) \cdot \underline{n}\, dS$$

For two functions f and g that are twice differentiable we can write:

$$\underline{a}(\underline{r}) = f(\underline{r}) \nabla g(\underline{r}) \quad \Rightarrow \quad \nabla \cdot \underline{a} = f \nabla^2 g + \nabla f \cdot \nabla g$$

and: $$\underline{a}'(\underline{r}) = g(\underline{r}) \nabla f(\underline{r}) \quad \Rightarrow \quad \nabla \cdot \underline{a}' = g \nabla^2 f + \nabla g \cdot \nabla f$$

$$\Rightarrow \quad \nabla \cdot (\underline{a} - \underline{a}') = f \nabla^2 g - g \nabla^2 f$$

$$\Rightarrow \quad \boxed{\int_V \left(f \nabla^2 g - g \nabla^2 f \right) dV = \oint_S \left(f \nabla g - g \nabla f \right) \cdot \underline{n}\, dS}$$

This is *Green's Second Identity.*

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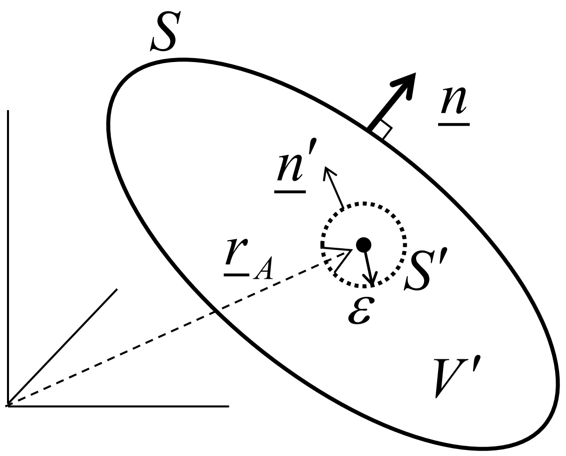
Kirchhoff Integral



For f substitute $\hat{p}(\underline{r})$, which is a solution of the Helmholtz equation:

$$\nabla^2 \hat{p} + \frac{\omega^2}{c^2} \hat{p} = 0$$

everywhere in the volume V.

For g substitute $\hat{G}(\underline{r}) = \dfrac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|}$, which is a solution of:

$$\nabla^2 \hat{G}(\underline{r}) + \frac{\omega^2}{c^2} \hat{G}(\underline{r}) = -\delta(\underline{r}-\underline{r}_A)$$

in the volume V', which is inside S, *but outside* the surface S' around the point-source at $\underline{r}_A$ inside V.

$$\Rightarrow \qquad \int_{V'} \left(\hat{p}\nabla^2\hat{G} - \hat{G}\nabla^2\hat{p}\right) dV' = \int_{V'} \left(-\hat{p}\frac{\omega^2}{c^2}\hat{G} + \hat{G}\frac{\omega^2}{c^2}\hat{p}\right) dV' \equiv 0$$

$$\Rightarrow \qquad \oint_S \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}\, dS - \oint_{S'} \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}'\, dS' = 0$$

Kirchhoff Integral

For the integral over S' we can write:

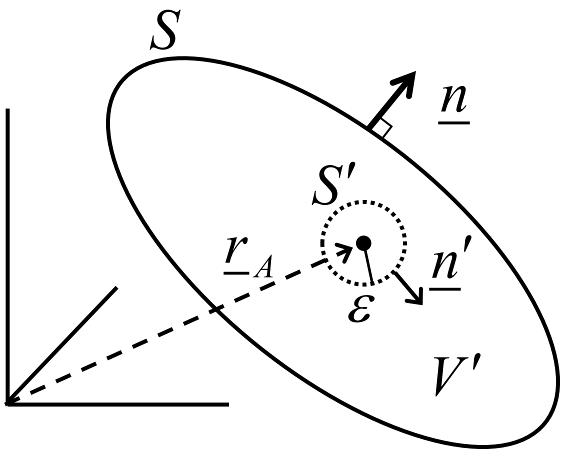
$$\lim_{\varepsilon\to 0}\oint_{S'}\left(\hat{p}\nabla\hat{G}-\hat{G}\nabla\hat{p}\right)\cdot\underline{n}'\,dS'$$

$$=\lim_{\varepsilon\to 0}\int_0^{\pi}\int_0^{2\pi}\left[\hat{p}\left(\underline{r}_A+\varepsilon\underline{n}'\right)\frac{\partial}{\partial\varepsilon}\left(\frac{e^{-i\omega\varepsilon/c}}{4\pi\varepsilon}\right)-\frac{e^{-i\omega\varepsilon/c}}{4\pi\varepsilon}\left(\nabla\hat{p}\cdot\underline{n}'\right)\right]\varepsilon^2\sin\vartheta\, d\varphi\, d\vartheta$$

$$=\lim_{\varepsilon\to 0}\int_0^{\pi}\int_0^{2\pi}\left[\hat{p}\left(\underline{r}_A\right)\left(-\frac{1}{\varepsilon^2}-\frac{i\omega}{c\varepsilon}\right)e^{-i\omega\varepsilon/c}\frac{1}{4\pi}\right]\varepsilon^2\sin\vartheta\, d\varphi\, d\vartheta$$

$$=-\,\hat{p}\left(\underline{r}_A\right)$$

$$\Rightarrow\quad \boxed{\hat{p}\left(\underline{r}_A\right)=-\oint_S\left(\hat{p}\nabla\hat{G}-\hat{G}\nabla\hat{p}\right)\cdot\underline{n}\,dS}$$



This is the *Kirchhoff integral.*

$$\hat{G}(\underline{r},\underline{r}_A,\omega)=\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi\left|\underline{r}-\underline{r}_A\right|}$$ is called the *Green's function.*

Kirchhoff Integral

$$\hat{p}\left(\underline{r}_A\right) = -\oint_S \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}\,dS \qquad \text{(point } A \text{ inside } S \text{)}$$

- The *Green's function* $\hat{G}(\underline{r},\omega)$ is the field of a point source located at $\underline{r}_A$ with delta-pulse wave-form.

- The field $\hat{G}$ does not co-exist with $\hat{p}$ in the same experiment. It is only introduced mathematically through Green's theorem.

- The Kirchhoff integral allows us to calculate the wave field $\hat{p}\left(\underline{r},\omega\right)$ at position $\underline{r}_A$, from recordings of $\hat{p}$ and $\left(\nabla\hat{p}\right)_n$ along any closed surface S around A.

- Application of the Kirchhoff integral can be cumbersome because:
  - We need recordings for both $\hat{p}$ and $\left(\nabla\hat{p}\right)_n$.
  - We need recordings along a closed surface.

- Under some limiting conditions there is a trick to be applied that circumvents both problems simultaneously.

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Rayleigh Integral

In the Kirchhoff integral:

$$\hat{p}(\underline{r}_A,\omega) = -\oint_S \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}\,dS$$

there is a degree of freedom, since for any function $\hat{\Gamma}(\underline{r},\omega)$ that satisfies:

$$\nabla^2\hat{\Gamma} + \frac{\omega^2}{c^2}\hat{\Gamma} = 0$$

everywhere inside S, we can write:

$$\hat{p}(\underline{r}_A) = -\oint_S \left[\hat{p}\nabla(\hat{G}+\hat{\Gamma}) - (\hat{G}+\hat{\Gamma})\nabla\hat{p}\right]\cdot\underline{n}\,dS$$

Obviously this is the case because:

$$\oint_S \left[\hat{p}\nabla\hat{\Gamma} - \hat{\Gamma}\nabla\hat{p}\right]\cdot\underline{n}\,dS = \int_V \left[\hat{p}\nabla^2\hat{\Gamma} - \hat{\Gamma}\nabla^2\hat{p}\right]dV = 0$$

Rayleigh Integral

We want to use the function Γ in the Kirchhoff integral:

$$\hat{p}\left(\underline{r}_A,\omega\right) = -\oint_S \left[\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right) - \left(\hat{G}+\hat{\Gamma}\right)\nabla\hat{p} \right] \cdot \underline{n}\, dS$$

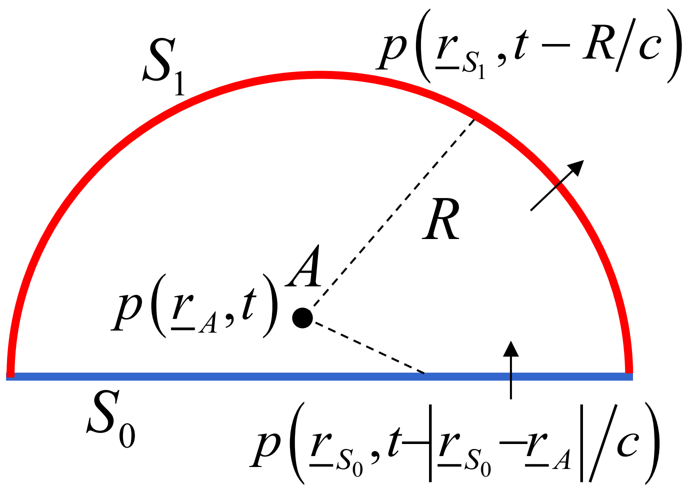
in such a way that either the term with $\hat{p}$ or the term with $\nabla\hat{p}$ vanishes over the relevant part of S.

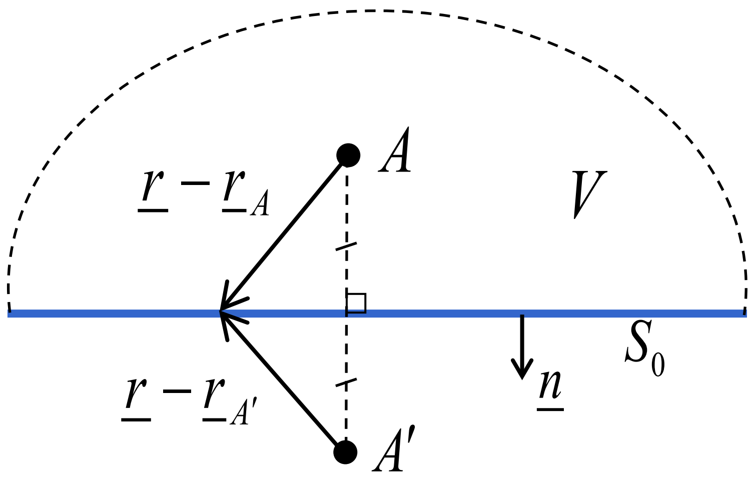
What the relevant part of S is depends on where the sources are that generated the wave field $\hat{p}$ and whether we want to predict forward or backward in time.

If there are sources in all directions from the point A , the whole closed surface S is relevant and there exists no suitable choice for $\hat{\Gamma}$ that simplifies the Kirchhoff integral.

Rayleigh Integral

$$\hat{p}(\underline{r}_A,\omega) = -\oint_S \left[\hat{p}\nabla(\hat{G}+\hat{\Gamma}) - (\hat{G}+\hat{\Gamma})\nabla\hat{p}\right]\cdot\underline{n}\,dS$$





$$\hat{p}(\underline{r}_A,\omega) = \oint_S \left[(\hat{G}+\hat{\Gamma})\nabla\hat{p}\right]\cdot\underline{n}\,dS \quad \leftarrow$$

$\nabla\left[\hat{G}(\underline{r})+\hat{\Gamma}(\underline{r})\right] = 0$ for all $\underline{r}\in S_0$ (Rayleigh I)

$$\hat{p}(\underline{r}_A,\omega) = -\oint_S \hat{p}\nabla(\hat{G}+\hat{\Gamma})\cdot\underline{n}\,dS \quad \leftarrow$$

$\hat{G}(\underline{r})+\hat{\Gamma}(\underline{r}) = 0$ for all $\underline{r}\in S_0$ (Rayleigh II)

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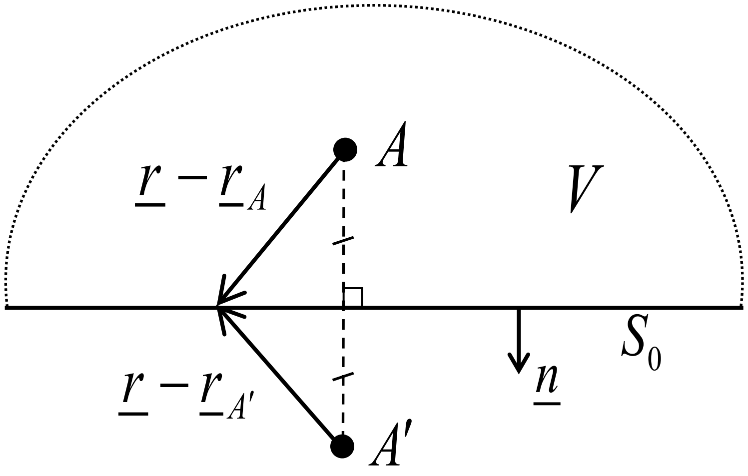
Rayleigh II Integral



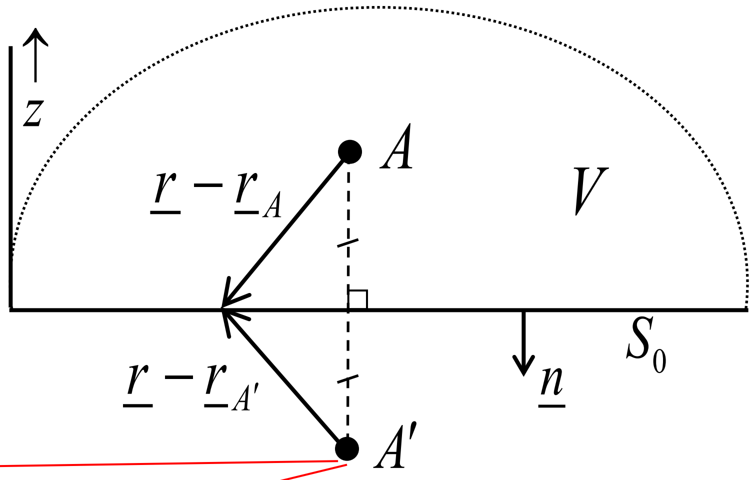
We have established that in the case that all the sources of the field $\hat{p}$ are below the plane S_0 , we only need to integrate over S_0 .

We now try to find a function $\hat{\Gamma}(\underline{r},\omega)$ that makes $\hat{G}+\hat{\Gamma}=0$ everywhere on S_0 .

Recalling that $\hat{G}$ is the wave field of a point source in point A , we can create a wave field $\hat{\Gamma}$ by putting a point-source with a negative source strength in the mirror point of A: A' .

This is legitimate because then the field $\hat{\Gamma}$ is not created by sources inside V and so satisfies the equation $\nabla^2\hat{\Gamma}+\left(\omega^2/c^2\right)\hat{\Gamma}=0$ everywhere inside V.

Rayleigh II Integral



If $\hat{G}+\hat{\Gamma}=0\Big|_{\vec{r}\in S_0}$, the Kirchhoff integral reduces to:

$$\hat{p}(\underline{r}_A)=-\int_{S_0}\left[\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right)\right]\cdot\underline{n}\,dS_0$$

with:

$$\hat{G}(\underline{r})=\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|} \quad \text{and} \quad \hat{\Gamma}(\underline{r})=-\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi|\underline{r}-\underline{r}_{A'}|}$$

where:

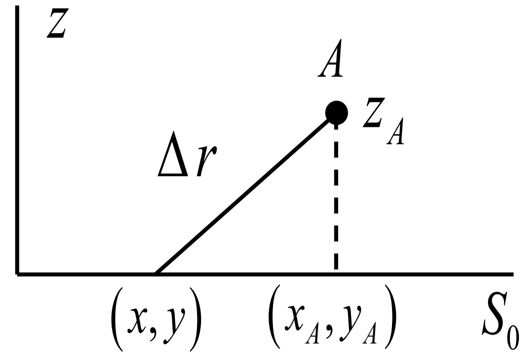
$$|\underline{r}-\underline{r}_A|=\sqrt{(x-x_A)^2+(y-y_A)^2+(z-z_A)^2}$$

$$|\underline{r}-\underline{r}_{A'}|=\sqrt{(x-x_A)^2+(y-y_A)^2+(z+z_A)^2}$$

On S_0 we have:

$$z=0 \quad , \quad |\underline{r}-\underline{r}_A|=|\underline{r}-\underline{r}_{A'}| \quad \text{and} \quad \nabla\left(\hat{G}+\hat{\Gamma}\right)\cdot\underline{n}=-\left[\frac{\partial}{\partial z}\left(\hat{G}+\hat{\Gamma}\right)\right]_{z=0}$$

Rayleigh II Integral



$$\frac{\partial}{\partial z}\left(\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|}\right) = -\frac{z-z_A}{|\underline{r}-\underline{r}_A|^3}\left(1+i\omega\frac{|\underline{r}-\underline{r}_A|}{c}\right)\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi}$$

$$\frac{\partial}{\partial z}\left(-\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi|\underline{r}-\underline{r}_{A'}|}\right) = \frac{z+z_A}{|\underline{r}-\underline{r}_{A'}|^3}\left(1+i\omega\frac{|\underline{r}-\underline{r}_{A'}|}{c}\right)\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi}$$

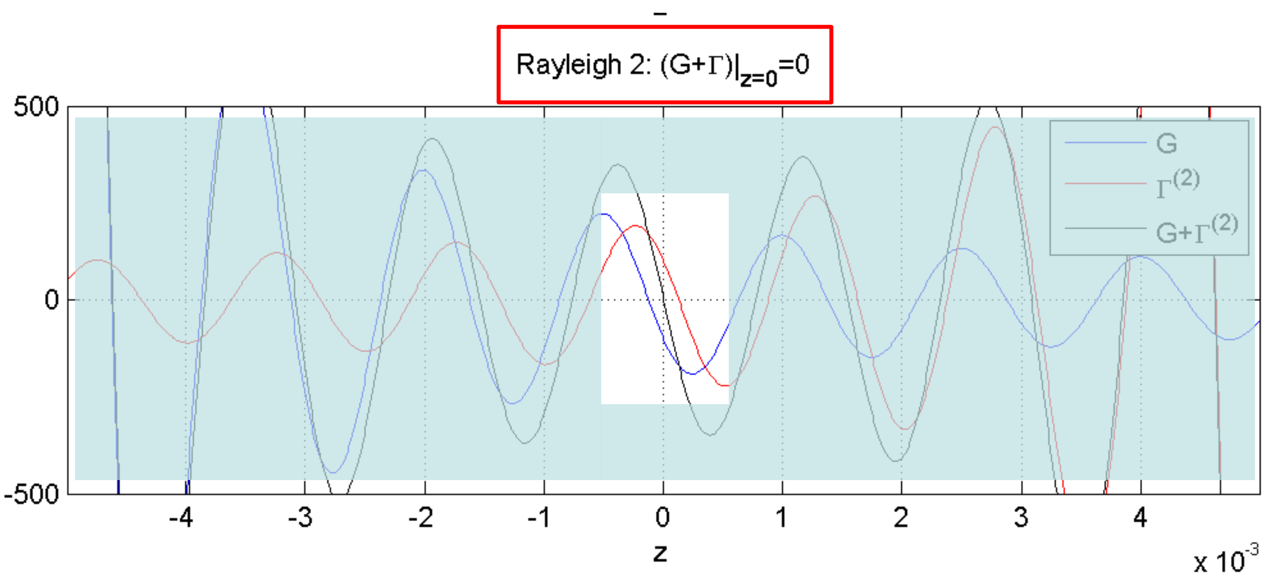
$$\left[\frac{\partial}{\partial z}\left(\hat{G}+\hat{\Gamma}\right)\right]_{z=0} = \frac{z_A\left(1+i\omega\Delta r/c\right)}{2\pi\,\Delta r^3}e^{-i\omega\Delta r/c}$$

$$\Rightarrow \quad \boxed{\hat{p}\left(\underline{r}_A,\omega\right) = \frac{z_A}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{p}\left(x,y,0;\omega\right)\left(1+i\omega\frac{\Delta r}{c}\right)\frac{e^{-i\omega\Delta r/c}}{\Delta r^3}\,dxdy}$$

$$\Delta r = \sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+z_A^2} \ .$$

This is the *Rayleigh II integral*.

Rayleigh Integral



Rayleigh I Integral

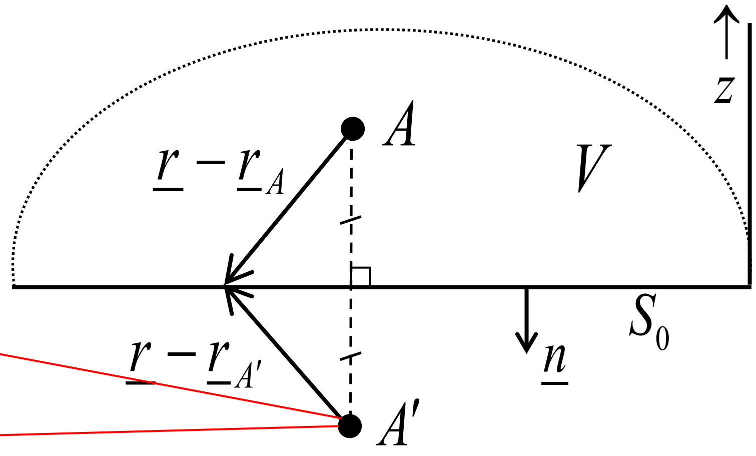
An alternative choice for $\hat{\Gamma}$ in the Kirchhoff integral

$$\hat{p}\left(\underline{r}_A,\omega\right) = -\oint_S \left[\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right) - \left(\hat{G}+\hat{\Gamma}\right)\nabla\hat{p}\right]\cdot \underline{n}\, dS$$

will cancel the term $\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right)$.

By choosing:

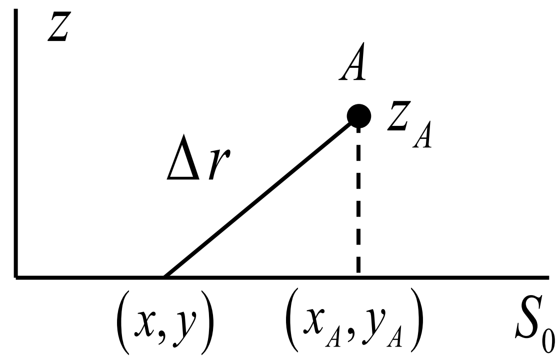
$$\hat{G}(\underline{r}) = \frac{e^{-i\omega\left|\underline{r}-\underline{r}_A\right|/c}}{4\pi\left|\underline{r}-\underline{r}_A\right|} \quad \text{and} \quad \hat{\Gamma}(\underline{r}) = \frac{e^{-i\omega\left|\underline{r}-\underline{r}_{A'}\right|/c}}{4\pi\left|\underline{r}-\underline{r}_{A'}\right|}$$



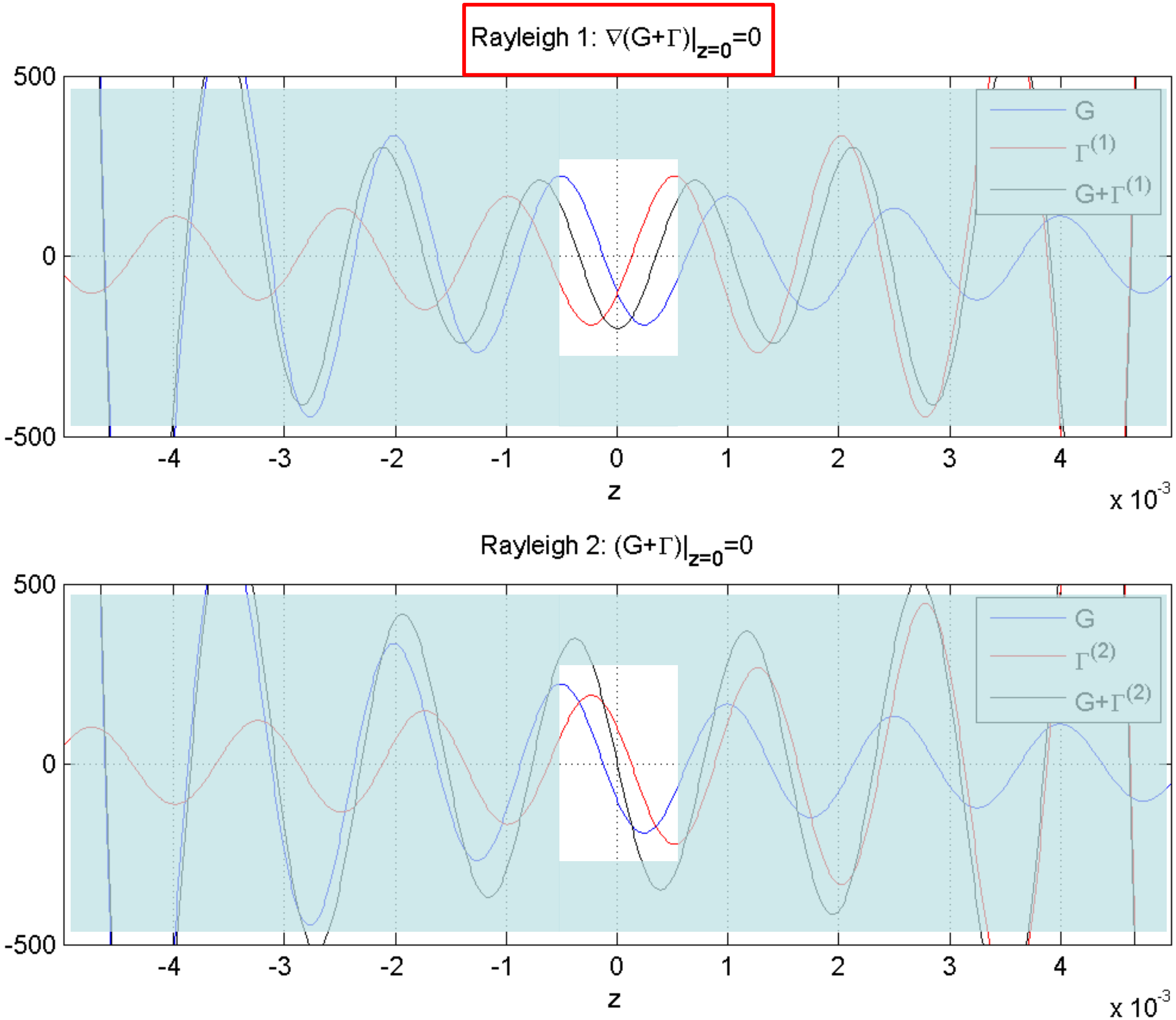
we obtain the *Rayleigh I integral*:

$$\hat{p}\left(\underline{r}_A,\omega\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\omega\Delta r/c}}{\Delta r}\left(\frac{\partial\hat{p}}{\partial z}\right)_{z=0} dxdy$$

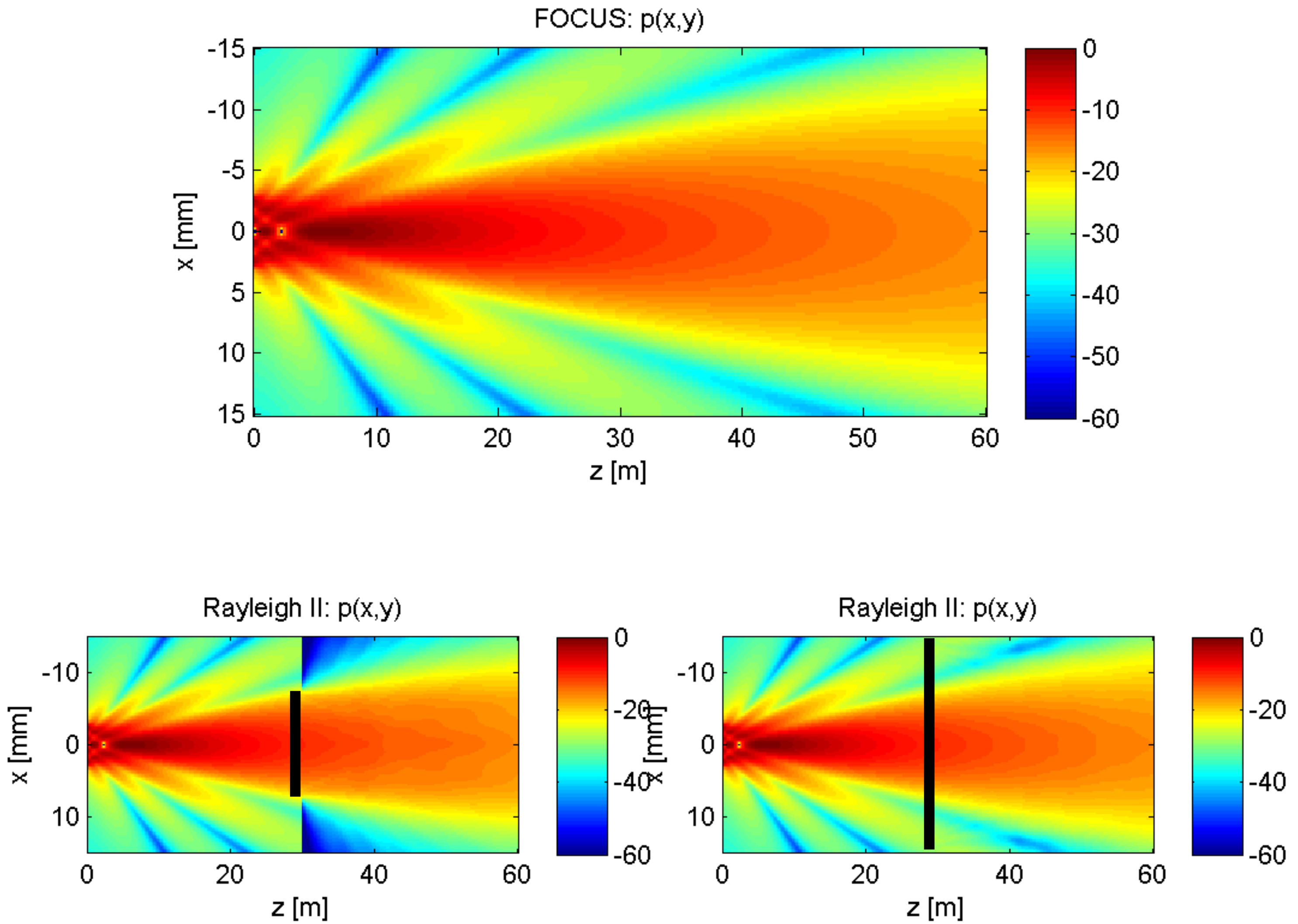
$$\Delta r = \sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+z_A^2}$$



Rayleigh Integral



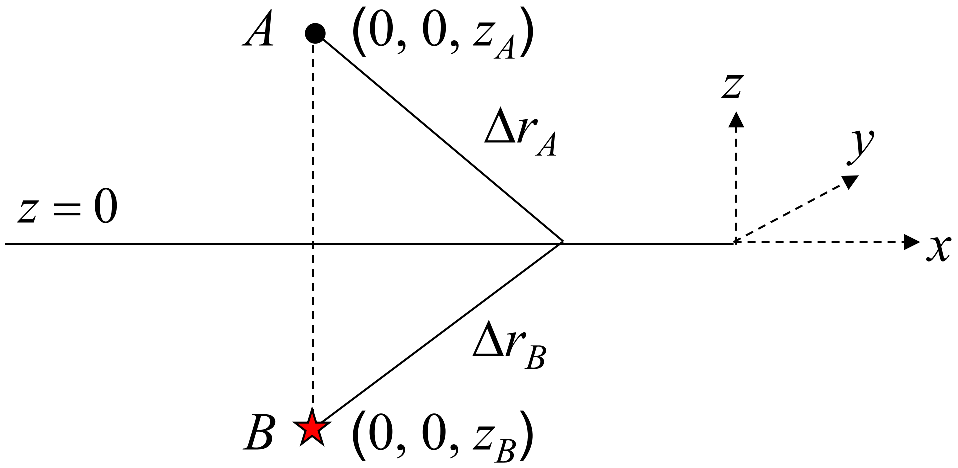
Rayleigh II Example



Exercise

Pointsource at point B. Wave-field recorded at plane $z = 0$.

Predict wave-field in point A, from observations along the plane $z = 0$.



$$\hat{p}(x,y,0;\omega) = \frac{e^{-i\omega\Delta r_B/c}}{\Delta r_B}\hat{W}(\omega) \quad , \quad \Delta r_B = \sqrt{x^2 + y^2 + z_B^2} \quad , \quad \Delta r_A = \sqrt{x^2 + y^2 + z_A^2}$$

Rayleigh II: $$\hat{p}(0,0,z_A;\omega) = \frac{z_A}{2\pi}\hat{W}(\omega)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\omega\Delta r_B/c}}{\Delta r_B} \frac{(1 + i\omega\Delta r_A/c)e^{-i\omega\Delta r_A/c}}{\Delta r_A^3} dxdy$$

Exercise (cont'd)

Switch to cylindrical coordinates: $\rho = \sqrt{x^2 + y^2}$, $x = \rho\cos\varphi$, $y = \rho\sin\varphi$

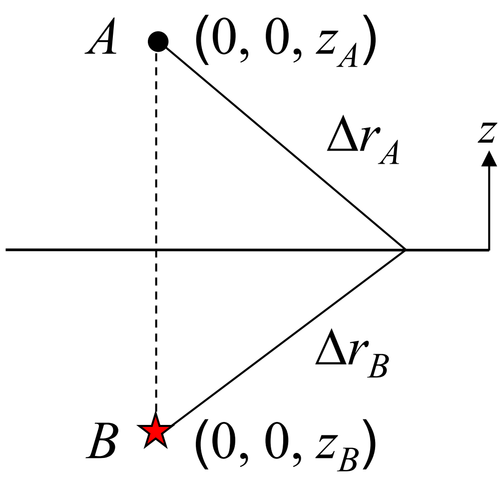
$$dxdy = \rho d\varphi d\rho \quad , \quad \Delta r_A = \sqrt{\rho^2 + z_A^2} \quad , \quad \Delta r_B = \sqrt{\rho^2 + z_B^2}$$

$$\hat{p}(0,0,z_A,\omega) = z_A \hat{W}(\omega) \int_0^\infty \rho \frac{1 + i\omega \Delta r_A / c}{\Delta r_B \Delta r_A^3} e^{-i\omega(\Delta r_A + \Delta r_B)/c} d\rho$$

$$= -z_A \hat{W}(\omega) \int_0^\infty \frac{\partial}{\partial\rho} \left[\frac{e^{-i\omega(\Delta r_A + \Delta r_B)/c}}{\Delta r_A (\Delta r_A + \Delta r_B)} \right] d\rho$$

$$= -z_A \hat{W}(\omega) \left[\frac{e^{-i\omega(\Delta r_A + \Delta r_B)/c}}{\Delta r_A (\Delta r_A + \Delta r_B)} \right]_{\rho=0}^{\rho=\infty}$$

$$= \hat{W}(\omega) \frac{e^{-i\omega(|z_A| + |z_B|)/c}}{|z_A| + |z_B|}$$



Rayleigh II Integral as Spatial Convolution

The Rayleigh II integral:

$$\hat{p}\left(x_A, y_A, z_A; \omega\right) = \frac{\Delta z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(x, y, z; \omega) \frac{1 + i\omega\left(\Delta r / c\right)}{\Delta r^3} e^{-i\omega(\Delta r / c)} dxdy$$

with: $\Delta r = \sqrt{\left(x_A - x\right)^2 + \left(y_A - y\right)^2 + \Delta z^2}$, $\Delta z = z_A - z > 0$

can be written as:

$$\hat{p}\left(x_A, y_A, z_A; \omega\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{Q}\left(x_A - x, y_A - y, \Delta z; \omega\right) \hat{p}(x, y, z; \omega)\, dxdy$$

with: $\hat{Q}\left(x, y, \Delta z; \omega\right) = \dfrac{\Delta z}{2\pi} \dfrac{\left(1 + \dfrac{i\omega}{c}\sqrt{x^2 + y^2 + \Delta z^2}\right)}{\left[x^2 + y^2 + \Delta z^2\right]^{3/2}} e^{-i(\omega/c)\sqrt{x^2 + y^2 + \Delta z^2}}$

$\hat{Q}\left(x, y, \Delta z; \omega\right)$ is the spatial convolution operator for forward extrapolation from the plane z to the plane $z_A = z + \Delta z$.

See also
George C. Sherman, "Application of the Convolution Theorem to Rayleigh's Integral Formulas," J. Opt. Soc. Am. 57, 546-547 (1967)

Helmholtz Equation in the (k_x, k_y)-domain

We define the double spatial Fourier transform of $\hat{p}(x, y, z; \omega)$:

$$\tilde{\hat{p}}(k_x, k_y; z; \omega) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} \hat{p}(x, y, z; \omega)\, dxdy$$

So, double Fourier transformation of the Helmholtz equation to the (k_x, k_y)-domain:

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + \frac{\omega^2}{c^2} \hat{p} = 0 \quad \Rightarrow \quad \frac{\partial^2 \tilde{\hat{p}}}{\partial z^2} + \left(\frac{\omega^2}{c^2} - k_x^2 - k_y^2 \right) \tilde{\hat{p}} = 0$$

with $k_z \equiv \sqrt{(\omega/c)^2 - k_x^2 - k_y^2}$, we get: $\boxed{\dfrac{\partial^2 \tilde{\hat{p}}}{\partial z^2} + k_z^2 \tilde{\hat{p}} = 0}$.

Helmholtz Equation in the (k_x,k_y)-domain

Let us consider a wave field $\tilde{\tilde{p}}\left(k_x,k_y,z;\omega\right)$, generated by sources below the $z=z_0$ plane.

Then the source-free Helmholtz equation:

$$\frac{\partial^2 \tilde{\tilde{p}}}{\partial z^2}+k_z^2\tilde{\tilde{p}}=0 \quad \text{with:} \quad k_z=\sqrt{\frac{\omega^2}{c^2}-\left(k_x^2+k_y^2\right)}$$

is valid for all $z \geq z_0$.

Since waves are travelling in the positive z direction only, the above differential equation in z is readily solved for $z \geq z_0$, by:

$$\boxed{\tilde{\tilde{p}}\left(k_x,k_y,z_0+\Delta z;\omega\right)=\tilde{\tilde{p}}\left(k_x,k_y,z_0;\omega\right)e^{-ik_z\Delta z}} \quad , \quad \Delta z>0$$

where $\tilde{\tilde{p}}\left(k_x,k_y,z_0\,;\omega\right)$ represents the double spatial Fourier transform of the observations made in the $z=z_0$ plane.

The factor $e^{-ik_z\Delta z}$ is the multiplicative forward extrapolation operator in the $\left(k_x,k_y,z\right)$-domain.

Evanescent Field

For $k_x^2 + k_y^2 > \frac{\omega^2}{c^2}$, we have

$$k_z = \sqrt{\frac{\omega^2}{c^2} - \left(k_x^2 + k_y^2\right)} = \pm i \sqrt{\left|\frac{\omega^2}{c^2} - \left(k_x^2 + k_y^2\right)\right|}$$

and:

$$e^{-ik_z \Delta z} = e^{\pm \Delta z \sqrt{\left|(\omega/c)^2 - \left(k_x^2 + k_y^2\right)\right|}}$$

As the plus sign would be physically unacceptable in the half space $\Delta z \geq 0$, we get:

$$\tilde{\tilde{p}}\left(k_x, k_y, z; \omega\right) = \tilde{\tilde{p}}\left(k_x, k_y, z_0; \omega\right) e^{-\Delta z \sqrt{\left|(\omega/c)^2 - \left(k_x^2 + k_y^2\right)\right|}} \quad , \quad k_x^2 + k_y^2 > \frac{\omega^2}{c^2}$$

Any energy in $\tilde{\tilde{p}}\left(k_x, k_y, z_0; \omega\right)$ for which $\left(k_x^2 + k_y^2\right) > \omega^2 / c^2$, dies out very quickly with increasing Δz . This is called the *evanescent* field.

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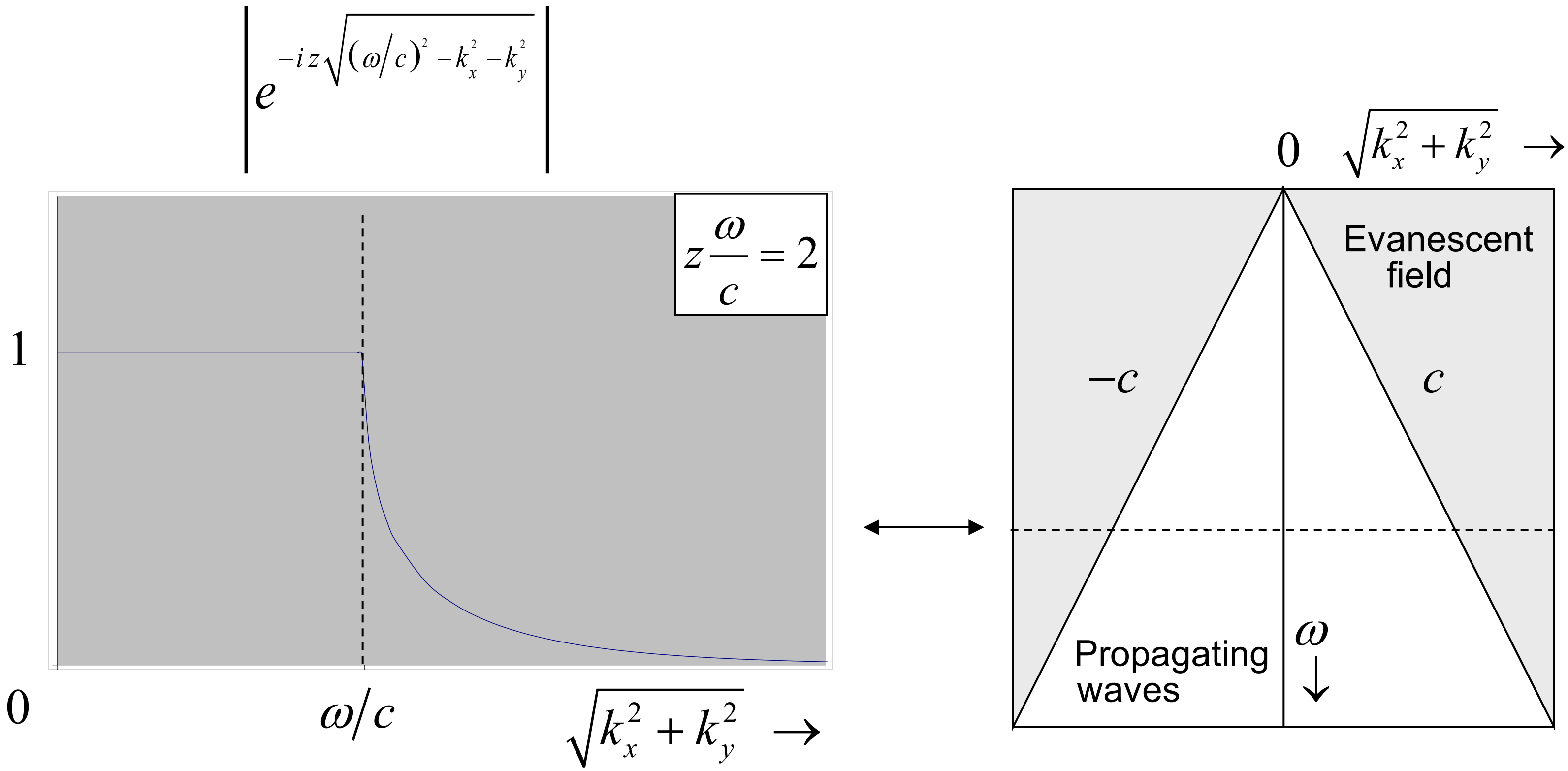
Evanescent Field



The evanescent field is observable only in 2-D plane-wave decompositions of wave-fields (remember $k_z \equiv \sqrt{(\omega/c)^2 - k_x^2 - k_y^2}$).

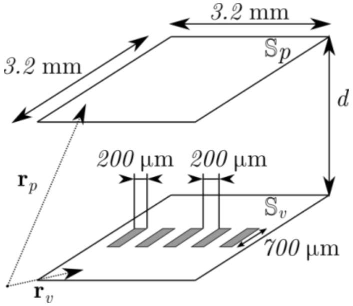
Example

This technique can be used to compute the velocity profile of an ultrasound transducer.

The pressure field at $z = a$ can be obtained from measurements at $z = 0$ as follows:

$$\tilde{\tilde{p}}\left(k_x, k_y, z = a\,; \omega\right) = \tilde{\tilde{p}}\left(k_x, k_y, z = 0\,; \omega\right) e^{-ik_z a} \quad ,$$

with $k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$, or $k_z = -i\sqrt{\left|\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right|}$ if $\frac{\omega^2}{c^2} < k_x^2 + k_y^2$.

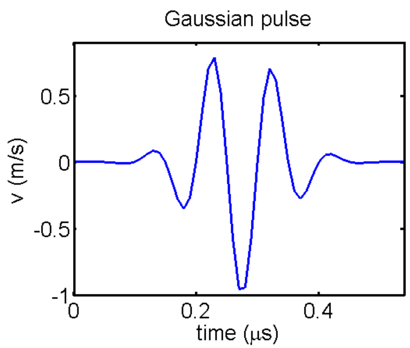


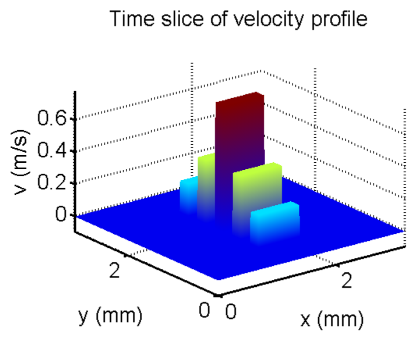
To obtain the field at $z = 0$ from measurements at $z = a$,

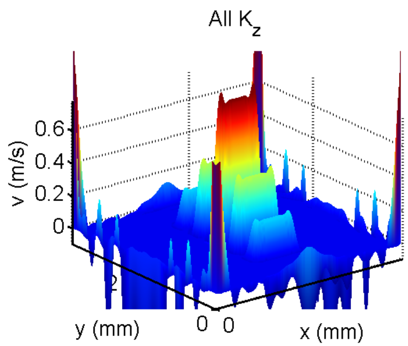
means we devide by $e^{-ik_z a}$, i.e.

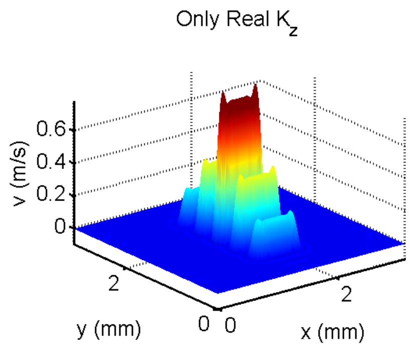
$$\tilde{\tilde{p}}\left(k_x, k_y, z = 0\,; \omega\right) = \tilde{\tilde{p}}\left(k_x, k_y, z = a\,; \omega\right) e^{+ik_z a}.$$

However, problems may arise in case for $k_z = -i\sqrt{\left|\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right|}$ if $\frac{\omega^2}{c^2} < k_x^2 + k_y^2$.



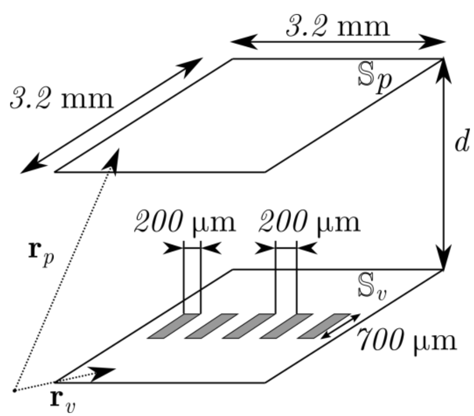


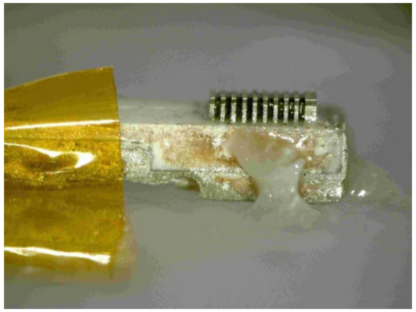


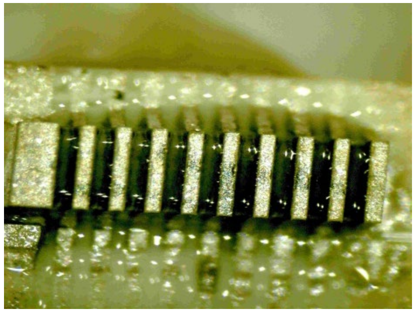


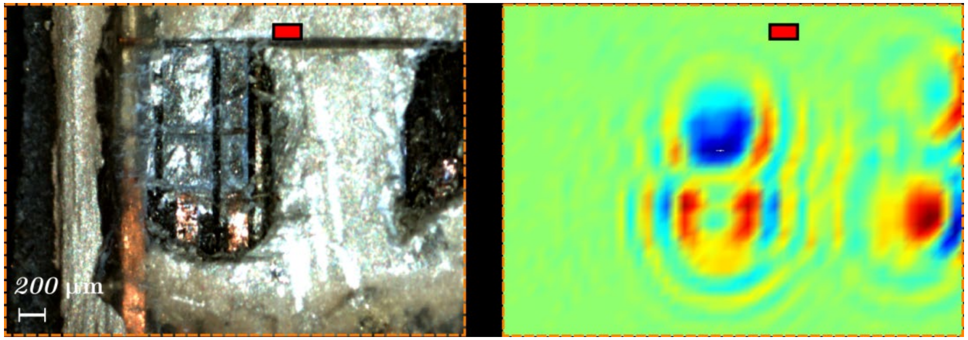
Example

- Reconstruction of the velocity profile of a damaged IVUS transducer.



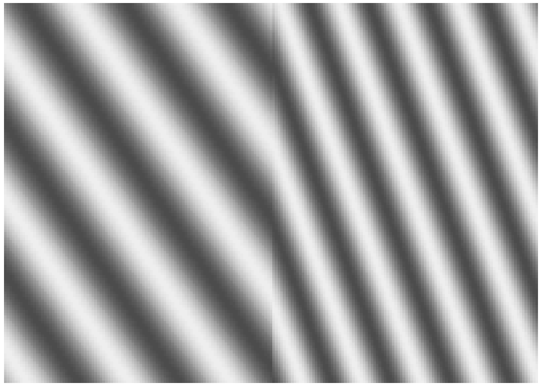




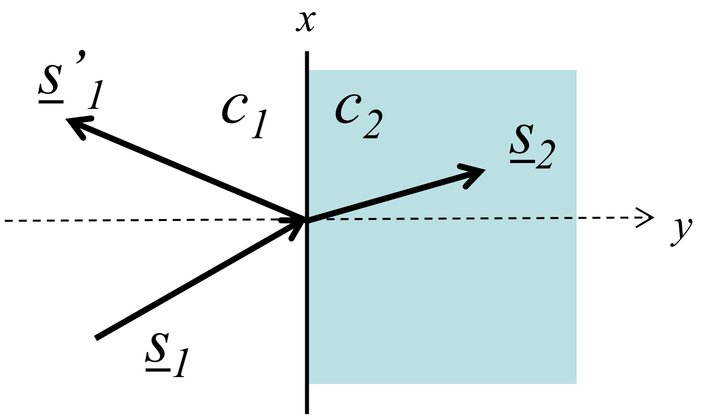


Day 2

Two Media - Boundary Conditions



Consider a plane wave traveling in the (x, y)-plane in the direction $\underline{s}_1$, with velocity $c_1 : \hat{p}_1(\underline{r},\omega) = F(\omega)e^{-i\omega \underline{s}_1 \cdot \underline{r}}$



If the field meets a boundary between two media with different speed of sound,

part of the field will be reflected: $\hat{p}'_1(\underline{r},\omega) = F(\omega)e^{-i\omega \underline{s}'_1 \cdot \underline{r}}$,

and part of field will be refracted: $\hat{p}_2(\underline{r},\omega) = F(\omega)e^{-i\omega \underline{s}_2 \cdot \underline{r}}$.

At $y = 0$, the following boundary conditions apply:

1) continuity of pressure: $\boxed{\hat{p}_1 + \hat{p}'_1 = \hat{p}_2}$;

2) continuity of normal component of the particle velocity: $\boxed{v_1^{\perp} + v'^{\perp}_1 = v_2^{\perp}}$.

Reflection and Refraction at a Plane Interface

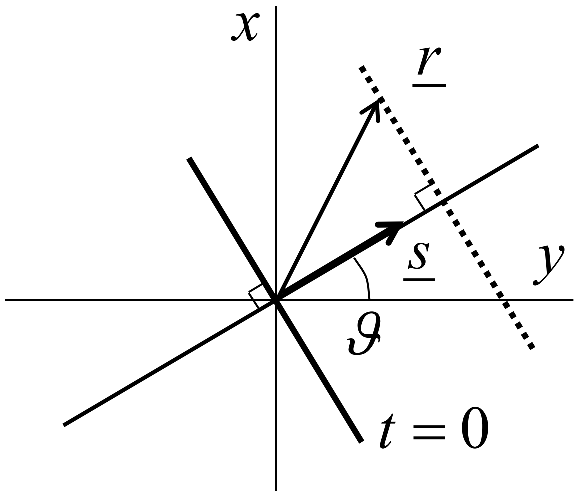
Consider a plane wave, travelling in the $(x, y)-$plane in the direction $\underline{s}$, with velocity c :

$$p(\underline{r},t) = f(t - \underline{s}\cdot\underline{r})$$

where:

$$\underline{s} = \frac{1}{c}(\sin\vartheta, \cos\vartheta)$$

is the slowness *vector*.



In the frequency domain:

$$P(\underline{r},\omega) = F(\omega)e^{-i\omega\underline{s}\cdot\underline{r}}$$

Reflection and Refraction at a Plane Interface

Consider a plane wave, travelling in the direction $\underline{\hat{s}}_1 = (\sin\vartheta_1, \cos\vartheta_1)$, incident on an interface between two media with properties ρ_{0_1}, c_1 and ρ_{0_2}, c_2 , under an angle ϑ_1 with the normal to the interface :

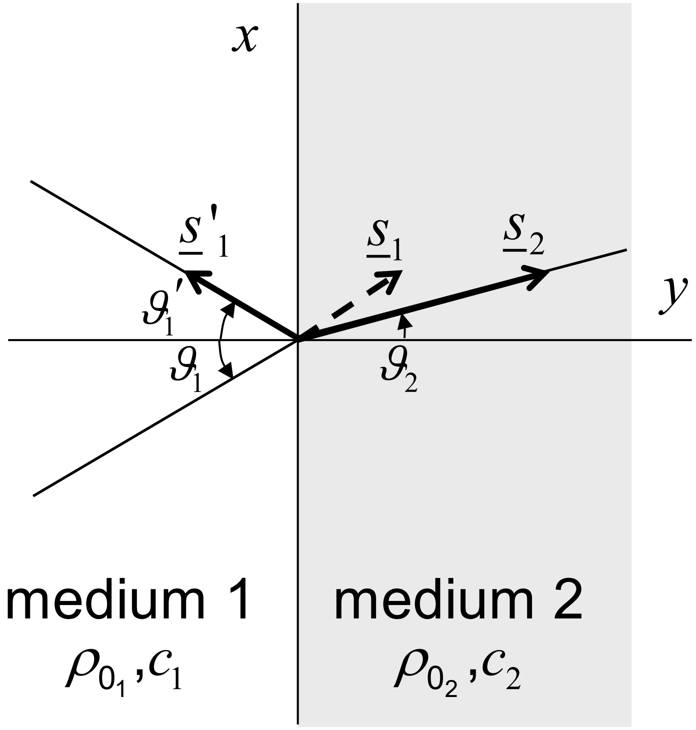
$$p_1(\underline{r},\omega) = F_1(\omega)\, e^{-i\frac{\omega}{c_1}(x\sin\vartheta_1 + y\cos\vartheta_1)}$$

There will be a reflected wave:

$$p_1'(\underline{r},\omega) = F_1'(\omega)\, e^{-i\frac{\omega}{c_1}(x\sin\vartheta_1' - y\cos\vartheta_1')}$$

and also a refracted wave:

$$p_2(\underline{r},\omega) = F_2(\omega)\, e^{-i\frac{\omega}{c_2}(x\sin\vartheta_2 + y\cos\vartheta_2)}$$



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Reflection and Refraction at a Plane Interface

At the interface, at $y = 0$, application of the boundary conditions yield

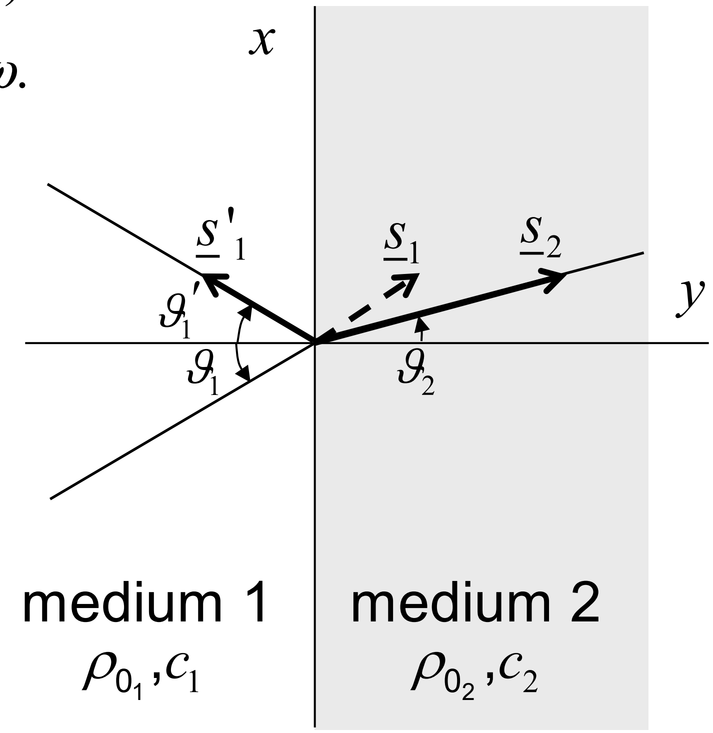
$$\begin{cases} p_1(x,0;\omega) + p_1'(x,0;\omega) = p_2(x,0;\omega) \\ v_{x;1}(x,0;\omega) + v_{x;1}'(x,0;\omega) = v_{x;2}(x,0;\omega) \end{cases}$$

These equations should hold for every x and ω.

Contiunuity of pressures results in:

$$F_1(\omega)e^{-i\frac{\omega}{c_1}x\sin\vartheta_1} + F_1'(\omega)e^{-i\frac{\omega}{c_1}x\sin\vartheta_1'}$$

$$= F_2(\omega)e^{-i\frac{\omega}{c_2}x\sin\vartheta_2}$$



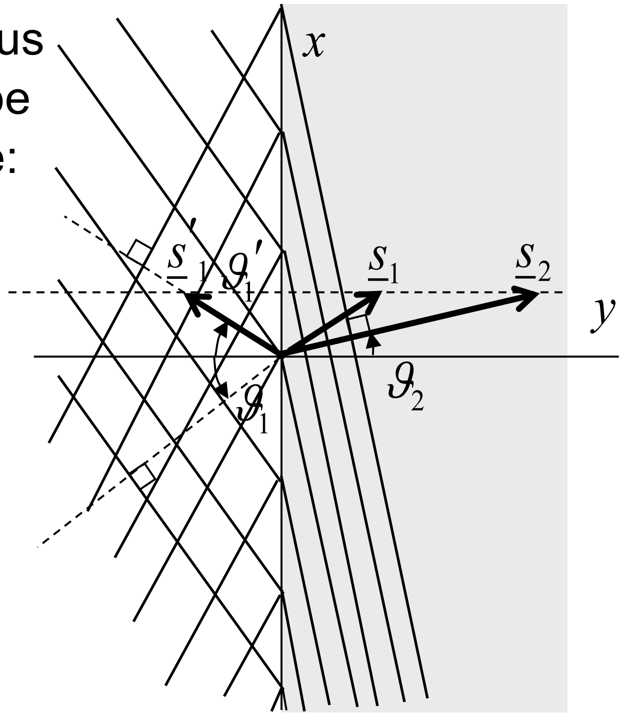
Reflection and Refraction at a Plane Interface

Continuity of pressure for all x and ω

can only be fulfilled if: $$\frac{\sin\vartheta_1}{c_1}=\frac{\sin\vartheta_1'}{c_1}=\frac{\sin\vartheta_2}{c_2}$$

which determines the reflection angle ϑ_1' and which gives us Snell's Law for the refraction angle ϑ_2. This law can also be expressed as *conservation of slowness* along the interface:

$$s_{x1}=s_{x1}'=s_{x2}$$



The boundary condition then reduces to:

$$F_1(\omega)+F_1'(\omega)=F_2(\omega) \quad \text{or:} \quad 1+R=T$$

with: $R=\dfrac{F_1'(\omega)}{F_1(\omega)}$ and: $T=\dfrac{F_2(\omega)}{F_1(\omega)}$

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Reflection and Refraction at a Plane Interface

Continuity of normal component of velocity:

$$v_1(x,0;\omega)\cos\vartheta_1 - v_1'(x,0;\omega)\cos\vartheta_1' = v_2(x,0;\omega)\cos\vartheta_2$$

Because the individual waves are all plane waves, we have:

$$v_1 = \frac{1}{\rho_{0_1}c_1}p_1 \quad , \quad v_1' = \frac{1}{\rho_{0_1}c_1}p_1' \quad \text{and} \quad v_2 = \frac{1}{\rho_{0_2}c_2}p_2$$

With $\vartheta_1 = \vartheta_1'$, we get:

$$\frac{\cos\vartheta_1}{\rho_{0_1}c_1}\left[F_1(\omega) - F_1'(\omega)\right] = \frac{1}{\rho_{0_2}c_2}F_2(\omega)\cos\vartheta_2$$

or:

$$\frac{\cos\vartheta_1}{\rho_{0_1}c_1}\left[1 - \frac{F_1'(\omega)}{F_1(\omega)}\right] = \frac{1}{\rho_{0_2}c_2}\frac{F_2(\omega)}{F_1(\omega)}\cos\vartheta_2$$

or:

$$\frac{\cos\vartheta_1}{\rho_{0_1}c_1}(1-R) = \frac{1}{\rho_{0_2}c_2}T\cos\vartheta_2$$

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Reflection and Refraction at a Plane Interface

From: $1+R=T$ and: $\dfrac{\cos\vartheta_1}{\rho_{0_1}c_1}(1-R)=\dfrac{1}{\rho_{0_2}c_2}T\cos\vartheta_2$

we can solve for R and T:

$$R=\frac{\rho_{0_2}c_2\cos\vartheta_1-\rho_{0_1}c_1\cos\vartheta_2}{\rho_{0_2}c_2\cos\vartheta_1+\rho_{0_1}c_1\cos\vartheta_2}\quad,\quad \cos\vartheta_2=\sqrt{1-\frac{c_2^2}{c_1^2}\sin^2\vartheta_1}$$

$$T=\frac{2\rho_{0_2}c_2\cos\vartheta_1}{\rho_{0_2}c_2\cos\vartheta_1+\rho_{0_1}c_1\cos\vartheta_2}$$

Note that $R\equiv\dfrac{F_1'(\omega)}{F_1(\omega)}$ and $T\equiv\dfrac{F_2(\omega)}{F_1(\omega)}$ are frequency independent.

With: $\rho_0 c\equiv Z$ we write:

$$R=\frac{Z_2\cos\vartheta_1-Z_1\cos\vartheta_2}{Z_2\cos\vartheta_1+Z_1\cos\vartheta_2}\quad\text{and:}\quad T=\frac{2Z_2\cos\vartheta_1}{Z_2\cos\vartheta_1+Z_1\cos\vartheta_2}$$

Critical Angle and Evanescent Waves

At the critical angle ϑ_c we have from Snell's law:

$$\sin\vartheta_2 = \frac{c_2}{c_1}\sin\vartheta_c = 1 \quad , \quad \cos\vartheta_2 = 0 \quad \Rightarrow \quad R = 1 \quad , \quad T = 2$$

All incident energy is reflected, but what about the transmitted amplitude $T = 2$?

For $\vartheta_1 > \vartheta_c$ we find from preservation of slowness in the x-direction:

$$s_{x2} = s_{x1} \quad \Rightarrow \quad \hat{s}_{x2} \equiv c_2 s_{x2} = c_2 \frac{\sin\vartheta_1}{c_1} > 1 \quad \Rightarrow \quad \hat{s}_{y2} = \sqrt{1-\hat{s}_{x2}^2} = \pm i\sqrt{\left|1-\hat{s}_{x2}^2\right|}$$

For the wave: $$P_2\left(\underline{r},\omega\right) = F_2\left(\omega\right)e^{-i\omega\left(s_{x2}x+s_{y2}y\right)}$$

we then get: $$P_2\left(\underline{r},\omega\right) = F_2\left(\omega\right)e^{-i\omega s_x x-\frac{\omega}{c_2}\left|\hat{s}_{y2}\right|y}$$

with: $s_x = s_{x1} = s_{x2}$ and: $\left|\hat{s}_{y2}\right| = \sqrt{\left|1-\hat{s}_{x2}^2\right|}$

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Critical Angle and Evanescent Waves

The wave: $p_2\left(\underline{r},\omega\right)=F_2\left(\omega\right)e^{-i\omega s_x x-\frac{\omega}{c_2}\left|\hat{s}_{y2}\right|y}$

with: $s_x = s_{x1} = s_{x2}$ and: $\left|\hat{s}_{y2}\right| = \sqrt{\left|1-\hat{s}_{x2}^2\right|}$ ($\left|\hat{s}_{x2}\right| > 1$)

propagates in the x-direction with slowness s_x . For the propagation velocity in the x-direction we get:

$$c_x \equiv \frac{1}{s_x} = \frac{1}{s_{x1}} = \frac{c_1}{\sin\vartheta_1} > c_1$$

but also:

$$c_x \equiv \frac{1}{s_x} = \frac{1}{s_{x2}} = \frac{c_2}{\hat{s}_{x2}} < c_2$$

In the y-direction this wave does not propagate, but it decays exponentially as $e^{-\omega\left|s_{y2}\right|y}$. Such a wave is called an *evanescent wave*.

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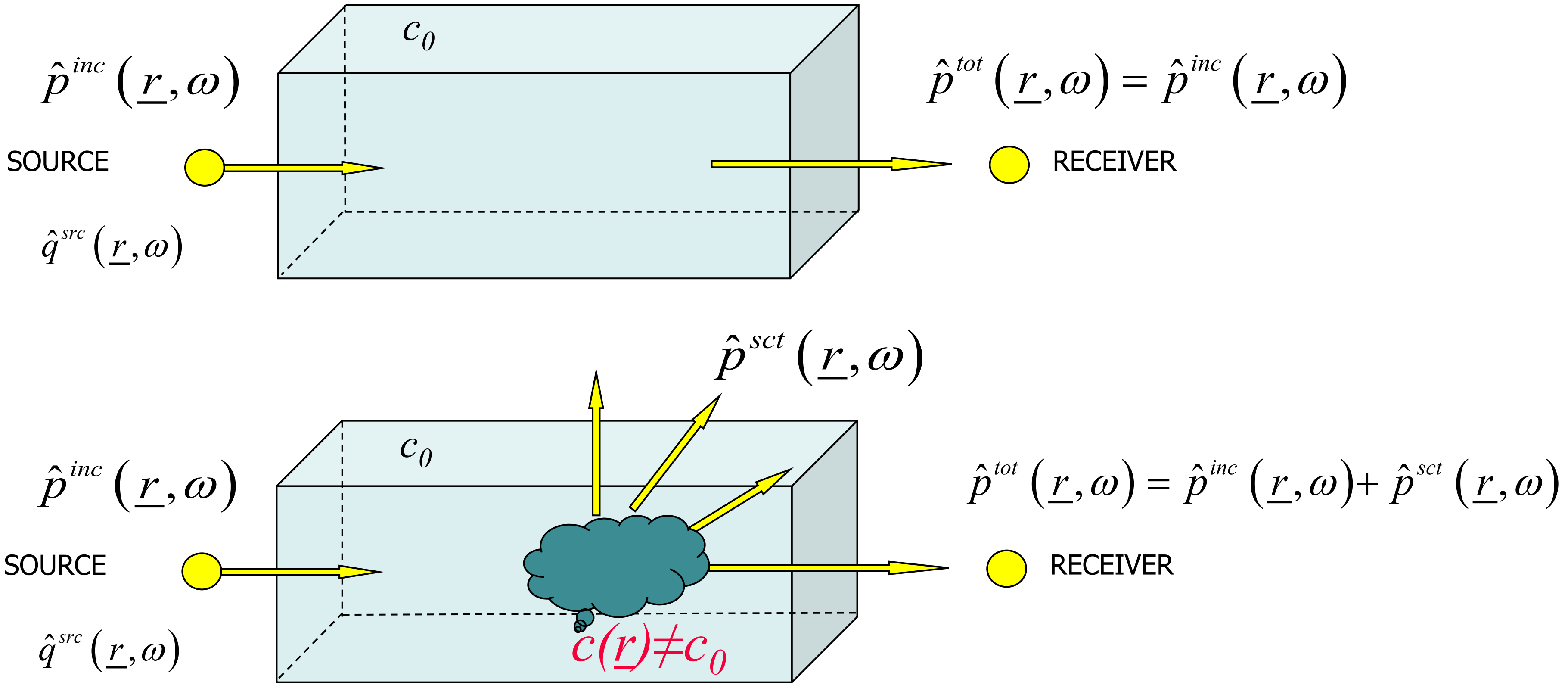
Heterogeneous Media

- Incident, scattered and total field
- Forward and inverse problem



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Heterogeneous Media – Field Equations

For heterogeneous media, the acoustic media parameters become spatially varying. Consequently, the resulting field equations will read

Hooke's law: $\nabla\bullet\underline{v}(\underline{r},t)+\kappa(\underline{r})\partial_t p(\underline{r},t)=q(\underline{r},t) \quad \Rightarrow \quad \begin{aligned}&\nabla\bullet\underline{v}(\underline{r},t)+\kappa_0\partial_t p(\underline{r},t)\\ &\quad =q(\underline{r},t)+\{\kappa_0-\kappa(\underline{r})\}\partial_t p(\underline{r},t)\end{aligned}$

Newton's law: $\nabla p(\underline{r},t)+\rho(\underline{r})\partial_t\underline{v}(\underline{r},t)=\underline{f}(\underline{r},t) \quad \Rightarrow \quad \begin{aligned}&\nabla p(\underline{r},t)+\rho_0\partial_t\underline{v}(\underline{r},t)\\ &\quad =\underline{f}(\underline{r},t)+\{\rho_0-\rho(\underline{r})\}\partial_t\underline{v}(\underline{r},t)\end{aligned}$

Combining the above field equations yield the following wave equation

$$\nabla^2 p(\underline{r},t)-\frac{1}{c_0^2}\partial_t^2 p(\underline{r},t)=-\underbrace{\{\rho_0\partial_t q(\underline{r},t)-\nabla\bullet\underline{f}(\underline{r},t)\}}_{S_{pr}(\underline{r},t)}-\underbrace{\{\rho_0(\kappa_0-\kappa(\underline{r}))\partial_t^2 p(\underline{r},t)-\nabla\bullet[\{\rho_0-\rho(\underline{r})\}\partial_t\underline{v}(\underline{r},t)]\}}_{S_{cs}(\underline{r},t)}$$

or

$$\nabla^2\hat{p}(\underline{r},\omega)+\frac{\omega^2}{c_0^2}\hat{p}(\underline{r},\omega)=-\{\rho_0 i\omega\hat{q}(\underline{r},\omega)-\nabla\bullet\underline{\hat{f}}(\underline{r},\omega)\}-\{-\rho_0(\kappa_0-\kappa(\underline{r}))\omega^2\hat{p}(\underline{r},\omega)-\nabla\bullet[\{\rho_0-\rho(\underline{r})\}i\omega\underline{\hat{v}}(\underline{r},\omega)]\}$$

Heterogeneous Media – Field Equations

Typically, spatial variations in the volume density of mass are neglected.

Consequently, the resulting field equations will read

Hooke's law: $\nabla\bullet\underline{v}(\underline{r},t)+\kappa(\underline{r})\partial_t p(\underline{r},t)=q(\underline{r},t) \quad\Rightarrow\quad \nabla\bullet\underline{v}(\underline{r},t)+\kappa_0\partial_t p(\underline{r},t) = q(\underline{r},t)+\{\kappa_0-\kappa(\underline{r})\}\partial_t p(\underline{r},t)$

Newton's law: $\nabla p(\underline{r},t)+\rho(\underline{r})\partial_t\underline{v}(\underline{r},t)=\underline{f}(\underline{r},t) \quad\Rightarrow\quad \nabla p(\underline{r},t)+\rho_0\partial_t\underline{v}(\underline{r},t) = \underline{f}(\underline{r},t)+\cancel{\{\rho_0-\rho(\underline{r})\}\partial_t\underline{v}(\underline{r},t)}$

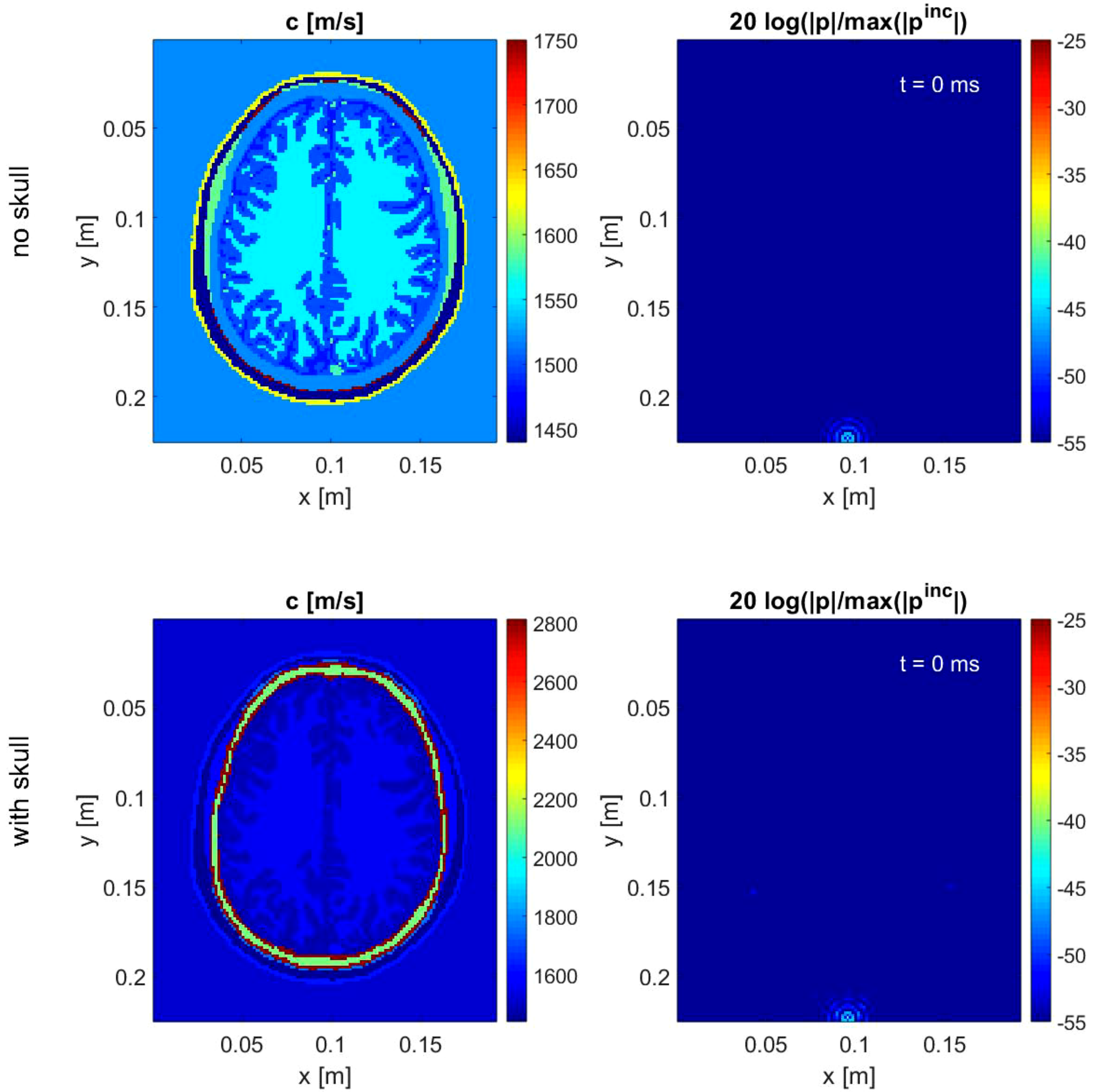
Combining the above field equations yield the following wave equation

$$\nabla^2 p(\underline{r},t)-\frac{1}{c_0^2}\partial_t^2 p(\underline{r},t) = -\underbrace{\{\rho_0\partial_t q(\underline{r},t)-\nabla\bullet\underline{f}(\underline{r},t)\}}_{S_{pr}(\underline{r},t)}-\underbrace{\left\{\left(\frac{1}{c_0^2}-\frac{1}{c^2(\underline{r})}\right)\partial_t^2 p(\underline{r},t)-\cancel{\nabla\bullet\left[\{\rho_0-\rho(\underline{r})\}\partial_t\underline{v}(\underline{r},t)\right]}\right\}}_{S_{cs}(\underline{r},t)}$$

or

$$\nabla^2\hat{p}(\underline{r},\omega)+\frac{\omega^2}{c_0^2}\hat{p}(\underline{r},\omega) = -\{\rho_0 i\omega\hat{q}(\underline{r},\omega)-\nabla\bullet\underline{\hat{f}}(\underline{r},\omega)\}-\left\{-\left(\frac{1}{c_0^2}-\frac{1}{c^2(\underline{r})}\right)\omega^2\hat{p}(\underline{r},\omega)-\cancel{\nabla\bullet\left[\{\rho_0-\rho(\underline{r})\}i\omega\underline{\hat{v}}(\underline{r},\omega)\right]}\right\}$$

Transcranial Ultrasound



Heterogeneous Media – Integral Equation

Green's function $\hat{G}(\underline{r},\omega) = \dfrac{e^{-i\omega|\underline{r}|/c_0}}{4\pi|\underline{r}|}$ represents the field generated by a Dirac-delta source.

Hence, the field generated by the primary sources $S_{pr}(\underline{r}',t)$ may be obtained by spatially convolving them with Green's function, hence

$$\hat{p}^{inc}(\underline{r},\omega) = \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\, S_{pr}(\underline{r}',\omega)\, dV(\underline{r}').$$

Based on the principle of superposition, one could argue that each contrast acts as a source generating an acoustic field. Adding all these fields together yields the following integral equation (Fredholm integral equation of the second kind)

$$\boxed{\hat{p}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\chi(\underline{r}')\omega^2\, \hat{p}(\underline{r}',\omega)\, dV(\underline{r}')}$$

with $\chi(\underline{r}') = \dfrac{1}{c^2(\underline{r}')} - \dfrac{1}{c_0^2}.$

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Heterogeneous Media – Integral Equation

- Forward problem: sources and contrast are known,
  total/actual field is unknown
  => linear problem

- Inverse problem: sources are known,
  total/actual field is known at the boundary,
  contrast and field in ROI is unknown.
  => non-linear problem

- Green's function is defined for the background medium, however there is a freedom to choose it heterogeneous or homogeneous.
  - Obvious choice is to choose a background for which we have an analytical expression of the Green's function (or the incident field).
  - For smooth varying media one could apply the WKBJ approximation
  - Green's function is singular at $|\underline{r}|=0$.

WKBJ = Wentzel – Kramers – Brillouin – Jeffreys

Born Approximation

If the contrast $\chi(\underline{r})$, or ω , or V are small enough, the integral equation:

$$\hat{p}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\,\chi(\underline{r}')\,\omega^2\,\hat{p}(\underline{r}',\omega)\,dV(\underline{r}')$$

can be linearised in the contrast χ by replacing $\hat{p}$ with $\hat{p}^{inc}$ on the right-hand side of the equation. We then get:

$$\hat{p}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\,\chi(\underline{r}')\,\omega^2\,\hat{p}^{inc}(\underline{r}',\omega)\,dV(\underline{r}')$$

from which $\hat{p}$ can be evaluated directly. This is called the *Born approximation.*

Neumann Series

The Born approximation can be seen as the first step in an iterative solution method.

The total field resulting from the Born approximation, $\hat{p}^{(1)}$, can be substituted on the right-hand side of the integral equation, to obtain the next iteration result $\hat{p}^{(2)}$, towards a solution of the full integral equation:

$$\hat{p}^{(2)}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\,\chi\left(\underline{r}'\right)\omega^2\,\hat{p}^{(1)}(\underline{r}',\omega)\,dV(\underline{r}')$$

The resulting values $\hat{p}^{(1)}$, $\hat{p}^{(2)}$,..., $\hat{p}^{(n)}$, form a series, which is called the *Neumann series*.

- For strong contrasts, the iterative scheme may not converge to the true solution resulting in a need for more advanced iterative solution methods such as conjugate gradient methods.

- For a limited number of configurations, an analytical solution exits.

Solution methods

The integral equation

$$\hat{p}(r,\omega) = \hat{p}^{inc}(\underline{r},\omega) + \int_{\underline{r}'\in D} \hat{G}(\underline{r}-\underline{r}',\omega)\, \chi\left(\underline{r}'\right) \omega^2\, \hat{p}(\underline{r}',\omega)\, dV(\underline{r}')$$

can be solved very efficiently, when it treated as a vector-matrix problem, i.e.

f=Lu

with $\mathrm{f} = \hat{p}^{inc}(\underline{r},\omega)$ the known incident field, $\mathrm{u} = \hat{p}(\underline{r},\omega)$ the unknown total field, and L the remaining integral operator (including identity matrix).

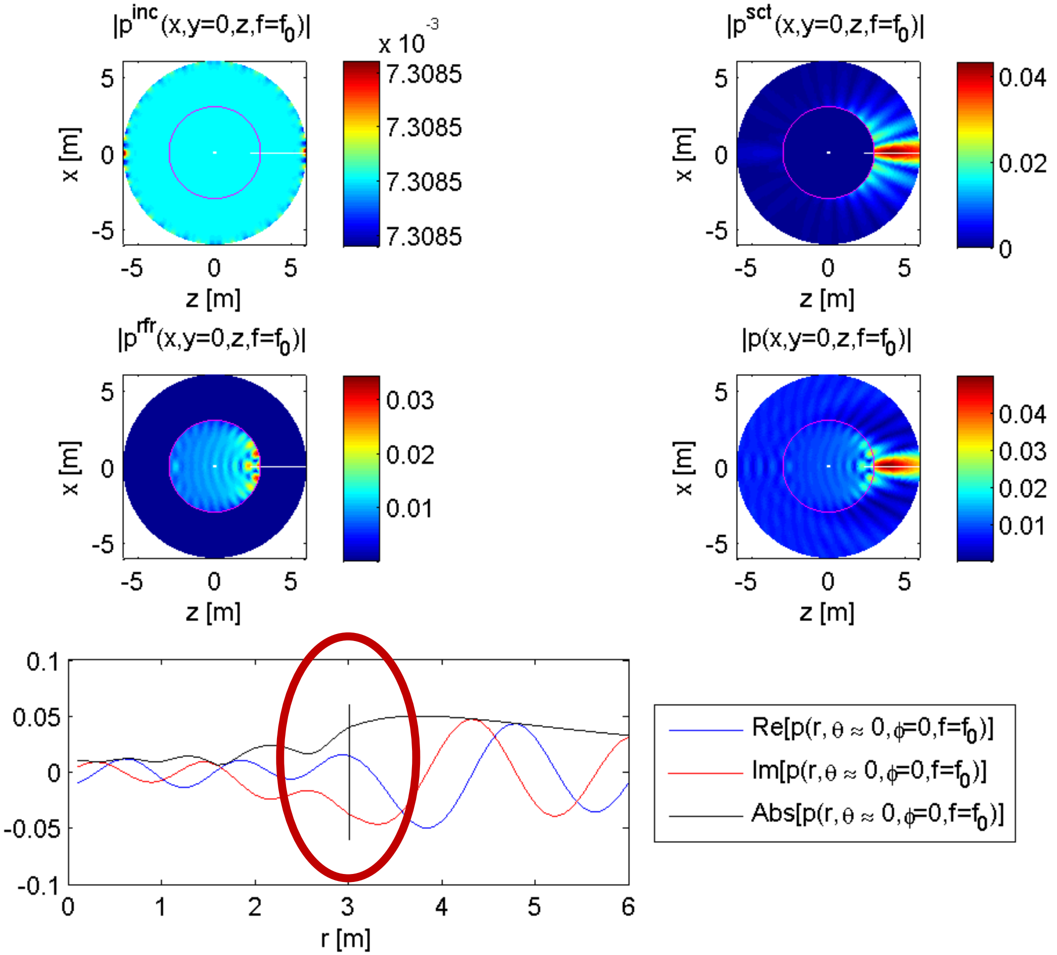
Such a vector-matrix problem can be solved efficiently using a conjugate gradient method. This method is based on the following scheme:

$$\mathrm{E_n} = \| \mathrm{r_n} \|^2 = \| \mathrm{f - Lu_n} \|^2$$
$$\mathrm{u_n} = \mathrm{u_{n-1}} + \alpha_{\mathrm{n}} \mathrm{d_n}$$
$$\mathrm{d_n} = \mathrm{L}^{\dagger} \mathrm{r_n}$$

Exact Solutions – Spherical Contrast

A plane wave scattering at a acoustical penetrable sphere in an homogeneous background medium maybe modelled using an integral equation formulation.

There also exists an exact solution for this problem.



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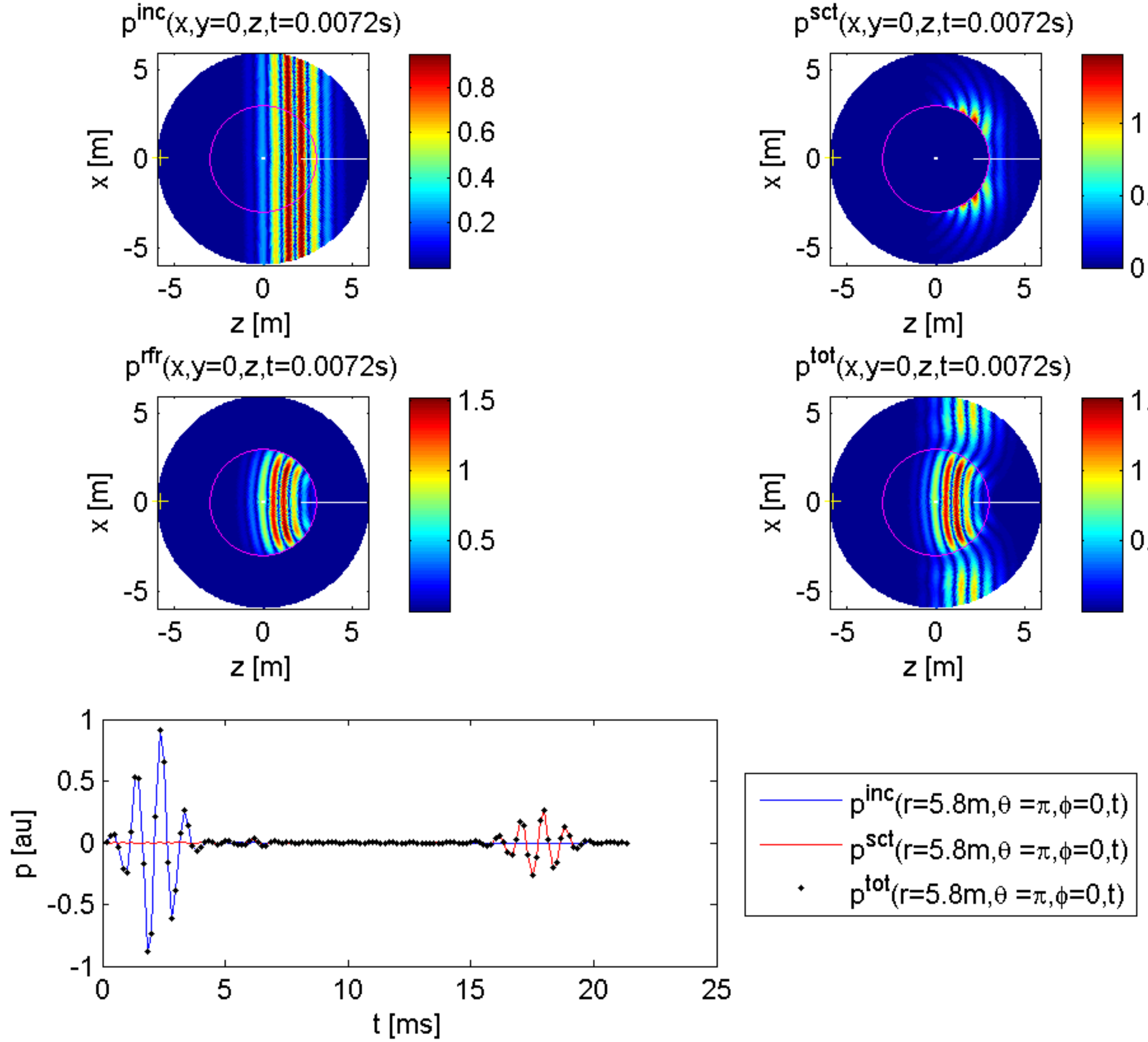
Exact Solutions – Spherical Contrast

By applying a Fourier transformation, time domain results are obtained.



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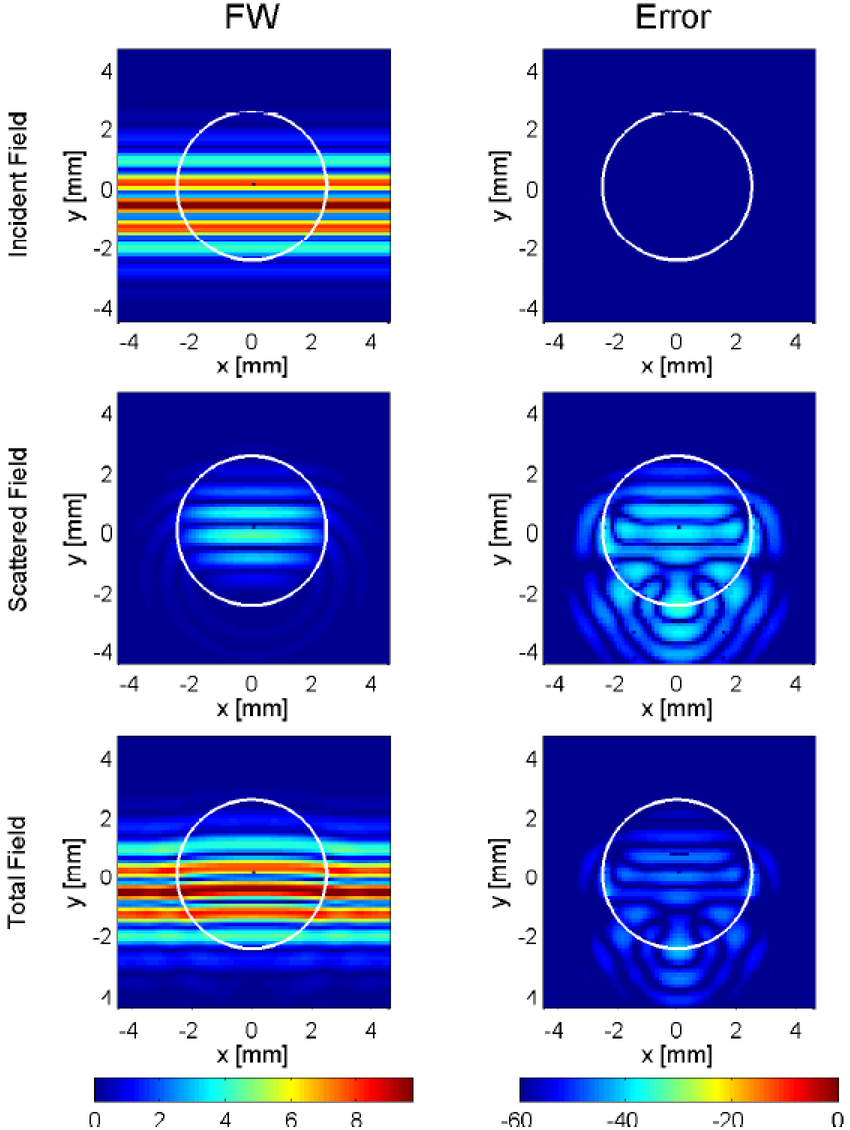
Exact Solutions – Spherical Contrast

Comparison of the obtained results shows that the integral equation formulation is rather accurate.



Finite Difference Time Domain (FDTD)

FDTD or Yee's method uses the time-domain field equations as a starting point. It has been first described by Courant, Friedrichs, and Lewy (CFL) in 1928. In 1966, Yee described the application of FDTD for solving Maxwell's curl equations using staggered grids in space and time.

The method uses finite difference rules to discretize the field equations; e.g.

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \Leftrightarrow \quad v_x^{q+\frac{1}{2}}\left[m,n,p\right] = v_x^{q-\frac{1}{2}}\left[m,n,p\right] - \frac{1}{\rho}\frac{\Delta t}{\delta}\left(p^q\left[m+1,n,p\right] - p^q\left[m,n,p\right]\right)$$

After discretising the equations and spatial domain, the starting conditions for the wave fields are defined. Finally, the four field quantities are solved in a leapfrog manner.

Finite Difference Time Domain (FDTD)

Pros

- It is intuitive, easy to understand and implement.
- FDTD is a time-domain technique and the response of the system over a wide range of frequencies can be obtained with a single simulation.

Cons

- Since FDTD requires the entire computational domain to be gridded and discretization must be sufficiently fine to resolve both the smallest wavelength and the smallest geometrical feature in the model. Also the time steps must be very small. This may lead to memory problems.
- Care must be taken to minimize errors introduced by boundaries (PML and ABC's).