Fundamentals of Ultrasound

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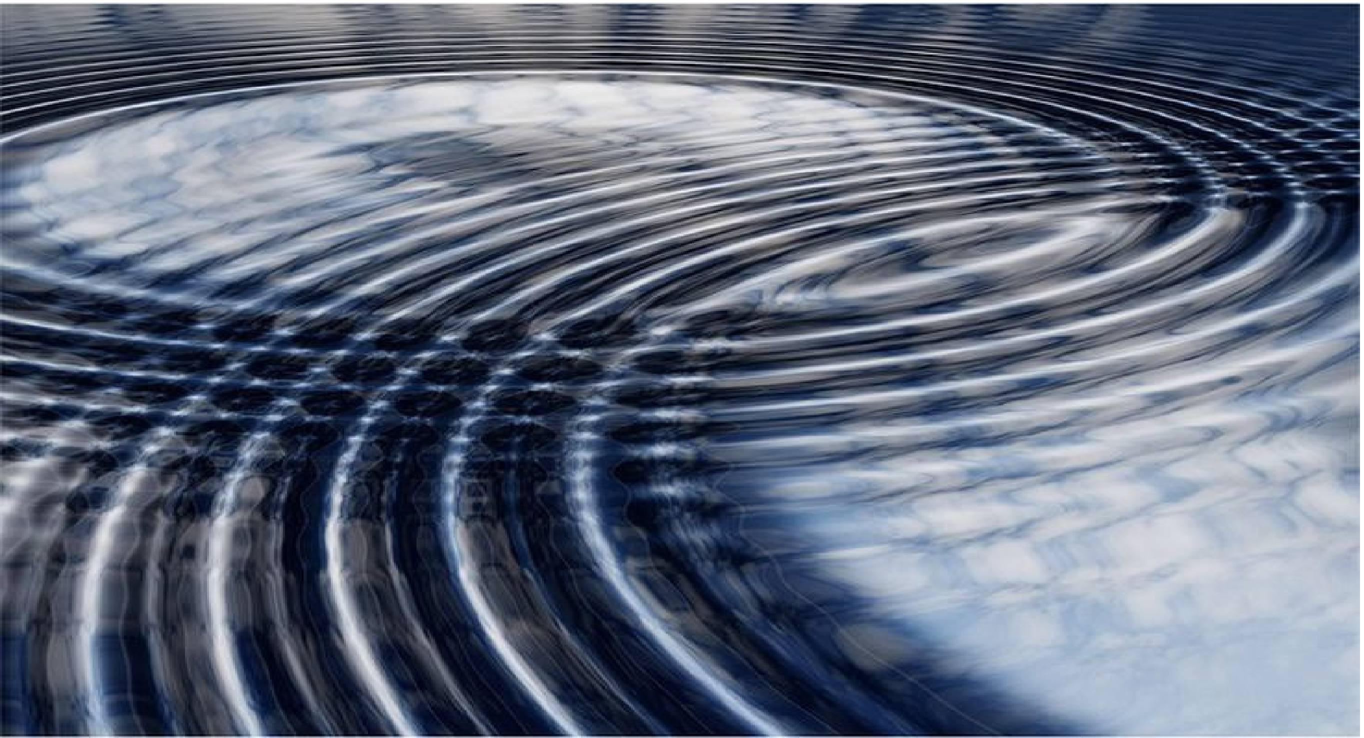

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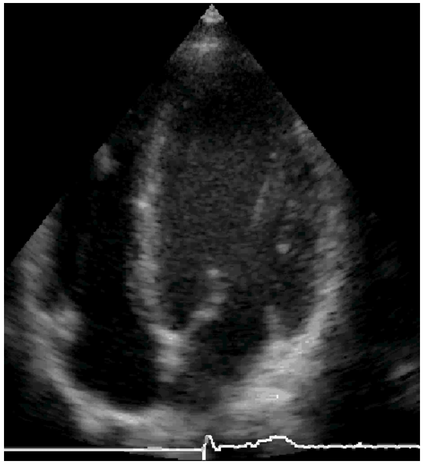
Introduction – Medical Ultrasound



[1] …
[2] Erasmus MC, Rotterdam, the Netherlands
[3] http://langeproductions.com/

Introduction – Nondestructive testing

[1] https://mtlabs.co.nz/services/non-destructive-testing/
[2] https://www.aerospacetestinginternational.com/features/introduction-to-non-destructive-testing.html

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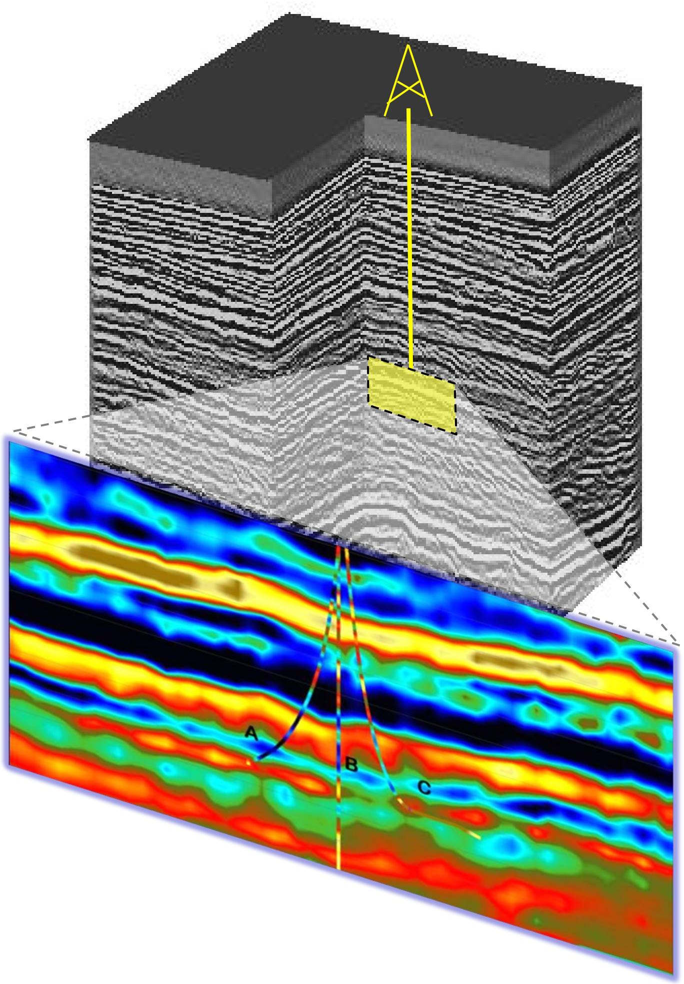
Introduction – Seismic Imaging



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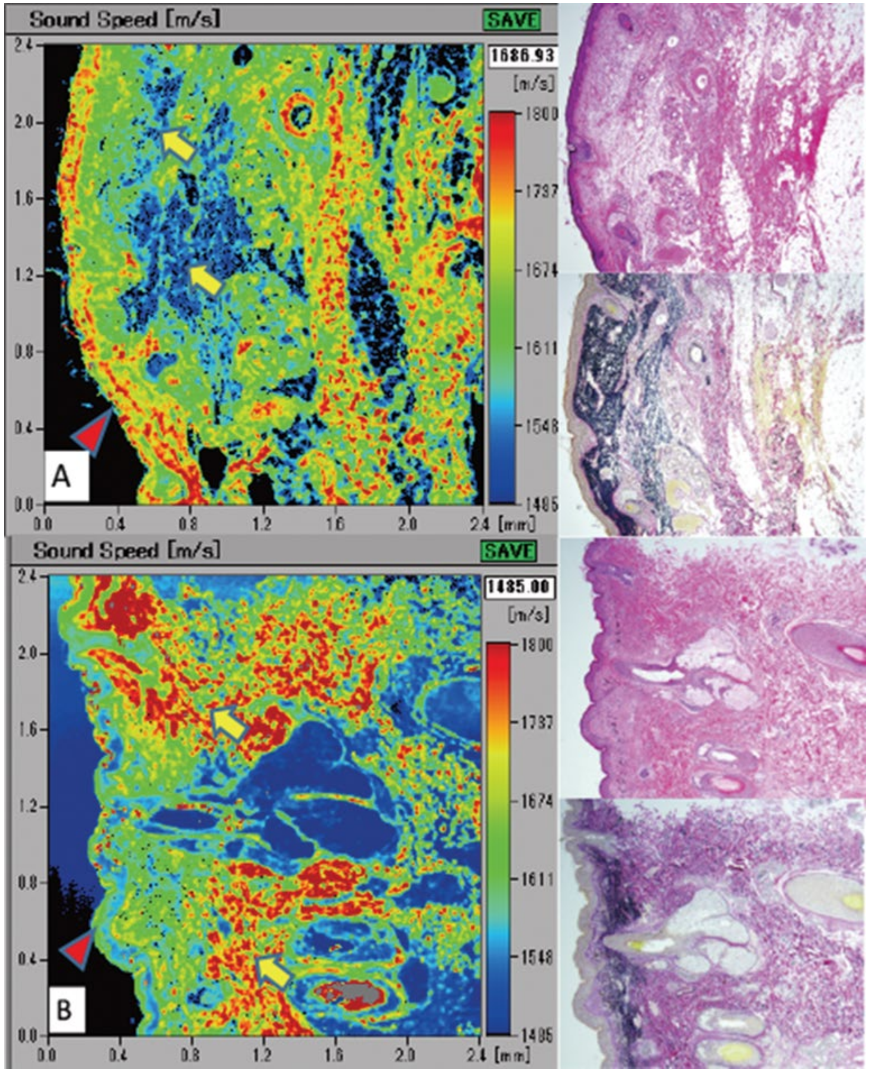
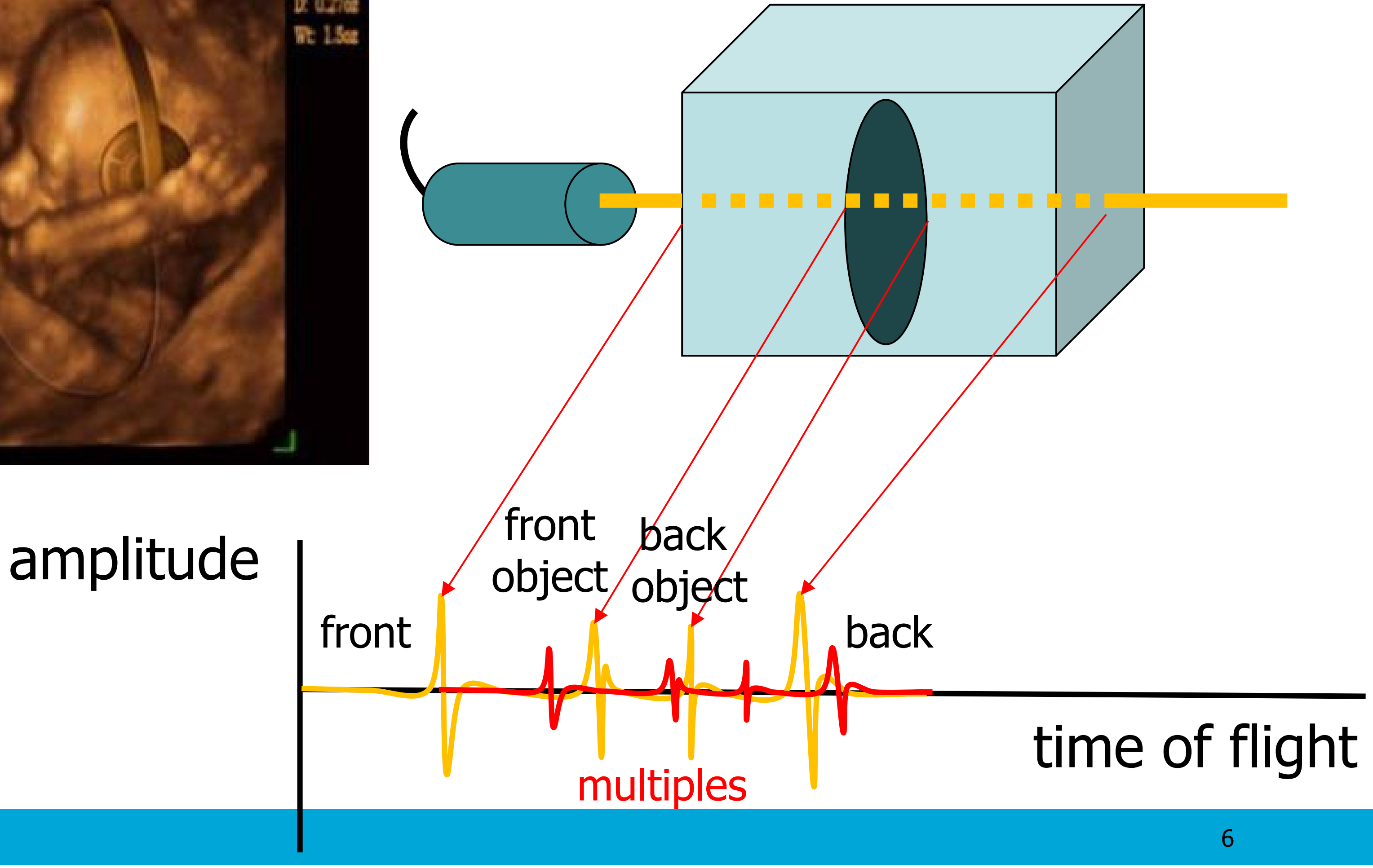
Introduction - Acoustic Microscopy

Comparative skin images of SOS



K. Miura, "Application of Scanning Acoustic Microscopy to Pathological Diagnosis," DOI: 10.5772/63405

Ultrasound scan



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Introduction – Applications Acoustical Imaging

- >50 MHz
- 10 MHz
- 1 MHz
- 100 kHz
- 10 kHz
- 1 kHz
- 100 Hz
- 10 Hz
- 1 Hz
- 0.1 Hz

Acoustic Microscopy

Laminated materials (e.g. airplanes) → 0.5 cm deep

Inspections of welds → 0.5-5 cm deep

Medical ultrasound → 0.2-20 cm deep

Ship wreck detection → 1-20 m deep

Near surface inspection → 5-500 m deep

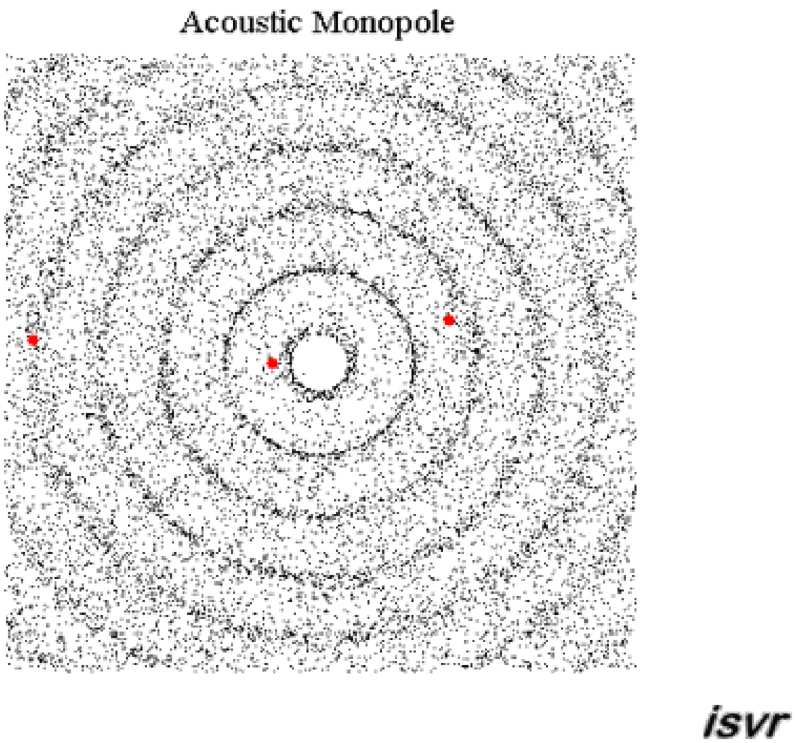
Oil & gas exploration → 0.5-5 km deep

Tectonic imaging → 2-25 km depth

Global seismology → full earth

Introduction

Field radiated by a point source / acoustic monopole [1]



[1] http://resource.isvr.soton.ac.uk/spcg/tutorial/tutorial/Tutorial_files/Web-basics-pointsources.htm

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Acoustic Field Equations

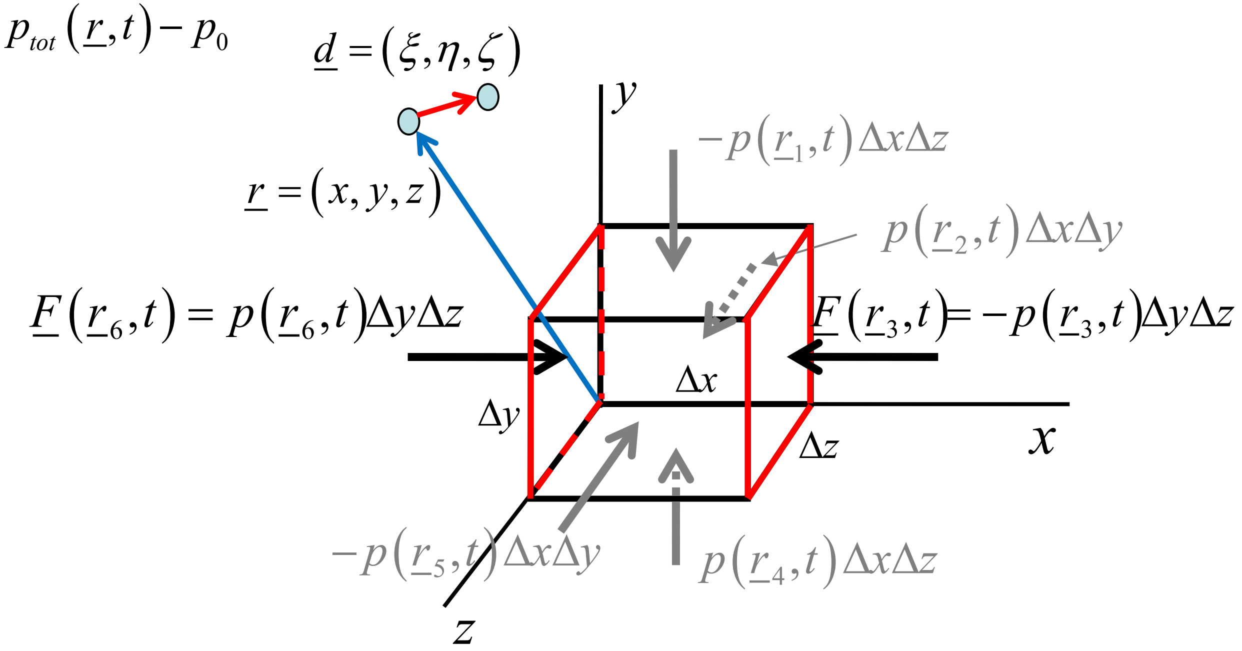
Volume: $\Delta V = \Delta x \Delta y \Delta z$

Location: $\underline{r} = (x, y, z)$

Displacement field: $\underline{d}(\underline{r},t) = \left[\xi(\underline{r},t), \eta(\underline{r},t), \zeta(\underline{r},t)\right]$

Velocity field: $\underline{v}(\underline{r},t) = \frac{d}{dt}\underline{d}(\underline{r},t)$

Pressure field: $p(\underline{r},t) = p_{tot}(\underline{r},t) - p_0$



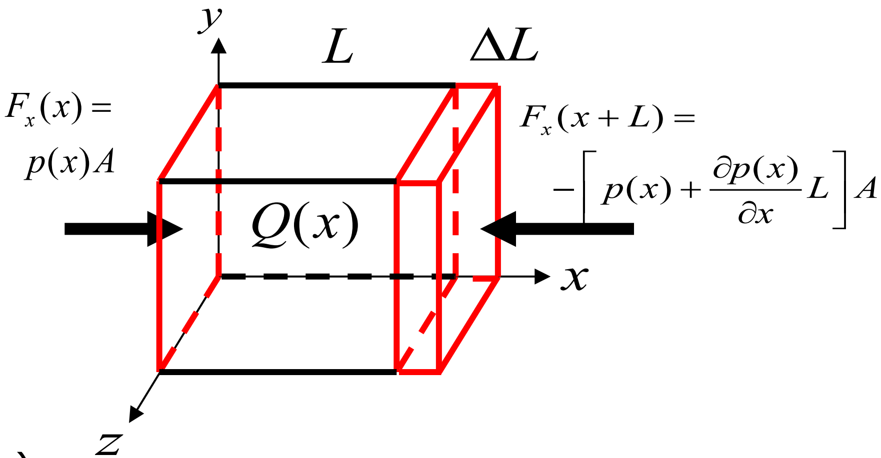
1-D Acoustic Field Equation – Hooke's Law

A fluid volume element ΔV is distorted by an excess pressure field $p(x,t)$ and volume injection $Q(x,t)$:

$$\Rightarrow \quad p_{tot}(x,t) = p_0 + p(x,t)$$

$$\Rightarrow \quad \Delta V \to \Delta V + A\Delta L$$

$$\Rightarrow \quad F = -k\frac{\Delta L}{L}$$ (Hooke's law: extension of a spring)



Amount of deformation depends on compressibility κ: $\frac{\Delta L}{L} = -\kappa p(x,t) + Q(x,t)$

Deformation can also be described by a displacement of the particle location $d_x(x,t)$: $\frac{\Delta L}{L} = \frac{\partial d_x(x,t)}{\partial x}$

$$\Rightarrow \quad -\kappa p(x,t) + Q(x,t) = \frac{\partial d_x(x,t)}{\partial x} \xrightarrow{\frac{1}{\kappa}\frac{d}{dt} = \cancel{\frac{1}{\kappa}v\cdot\nabla} + \frac{1}{\kappa}\frac{\partial}{\partial t}} \boxed{\frac{\partial p(x,t)}{\partial t} = -\frac{1}{\kappa}\frac{\partial v_x(x,t)}{\partial x} + \frac{1}{\kappa}q(x,t)}$$

Note that two linearizations are made: $\frac{d}{dt} = \cancel{v\cdot\nabla} + \frac{\partial}{\partial t}$ and $\kappa(p) = \kappa(p_0) + \cancel{(p-p_0)\left.\frac{\partial\kappa(p)}{\partial p}\right|_{p=p_0}}$

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3-D Acoustic Field Equation – Hooke's Law

Changing from one to three dimensions means that the volume will change in three dimensions and that the particles may move in the ξ, η and ζ – direction, hence

$$\frac{\delta\Delta V}{\Delta V} = \frac{\{\Delta x + \delta\xi\}\{\Delta y + \delta\eta\}\{\Delta z + \delta\zeta\} - \Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z} = \frac{\delta\xi\Delta y\Delta z + \delta\eta\Delta x\Delta z + \delta\zeta\Delta x\Delta y}{\Delta x\Delta y\Delta z} + O(\ldots) \approx \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}$$

$$\Rightarrow \quad \boxed{\frac{\delta\Delta V}{\Delta V} = \nabla\cdot\underline{d}}$$

Consequently, Hooke's law will change into

$$\boxed{\frac{\partial p(\underline{r},t)}{\partial t} = -\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t) + \frac{1}{\kappa}q(\underline{r},t)}$$

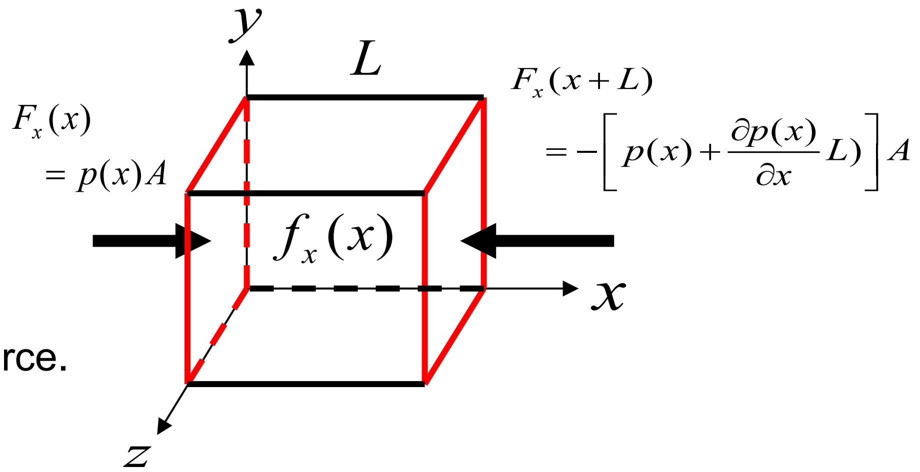
1-D Acoustic Field Equation – Newton's Law

With $p(x+L) = p(x) + \frac{\partial p}{\partial x} L$

we find for the net excess force ΔF on ΔV:

$$\Delta F = -A\left(\frac{\partial p(x,t)}{\partial x} L\right) + f_x(x,t)\Delta V$$

with $f_x(x,t)$ the volume source density of volume force.



Newton's Law, $F = ma$, for a fixed mass element $m = \Delta V \rho$ and $a = \frac{d\,v_x}{dt} = \cancel{v_x \frac{\partial v_x}{\partial x}} + \frac{\partial v_x}{\partial t} = \frac{\partial v_x}{\partial t}$ now reads:

$$\underbrace{-\Delta V \frac{\partial p(x,t)}{\partial x} + f_x(x,t)\Delta V}_{F} = \underbrace{\Delta V \rho}_{m} \underbrace{\frac{\partial v_x(x,t)}{\partial t}}_{a}$$

or:

$$\boxed{\frac{\partial p(x,t)}{\partial x} = -\rho \frac{\partial v_x(x,t)}{\partial t} + f_x(x,t)}$$

in 3-D the Newton's law reads:

$$\boxed{\nabla p(\underline{r},t) = -\rho \frac{\partial \underline{v}(\underline{r},t)}{\partial t} + \underline{f}(\underline{r},t)}$$

Note that two linearizations are made: $\frac{d}{dt} = \cancel{\underline{v}\cdot\nabla} + \frac{\partial}{\partial t}$ and $\rho(p) = \rho(p_0) + \cancel{(p-p_0)\left.\frac{\partial \rho(p)}{\partial p}\right|_{p=p_0}}$

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Acoustic Field Equations

The obtained acoustic field equations read

Hooke's law: $$\frac{\partial p(\underline{r},t)}{\partial t} = -\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t) + \frac{1}{\kappa}q(\underline{r},t)$$ (equation of deformation)

Newton's law: $$\nabla p(\underline{r},t) = -\rho\frac{\partial\underline{v}(\underline{r},t)}{\partial t} + \underline{f}(\underline{r},t)$$ (equation of motion)

This set of equations show large similarities with Maxwel Equations

$$\nabla\cdot\underline{E} = \frac{1}{\varepsilon_0}\rho_f \qquad \nabla\cdot\underline{B} = 0$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t} \qquad \nabla\times\underline{B} = \mu\sigma\underline{E} + \mu\varepsilon\frac{\partial\underline{E}}{\partial t}$$

Wave Equation

The acoustic field equations may be combined to obtain (in the absense of sources):

a) a scalar wave equation for the pressure field:

$$\left.\begin{aligned} \rho\kappa\frac{\partial}{\partial t}\left[\frac{\partial\, p(\underline{r},t)}{\partial t}\right] &= \rho\kappa\frac{\partial}{\partial t}\left[-\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t)\right] \\ \nabla\cdot\left[-\nabla p\left(\underline{r},t\right)\right] &= \nabla\cdot\left[\rho\frac{\partial\underline{v}\left(\underline{r},t\right)}{\partial t}\right] \end{aligned}\right\} \Rightarrow \boxed{\nabla^2 p\left(\underline{r},t\right) - \rho\kappa\frac{\partial^2\, p(\underline{r},t)}{\partial t^2} = 0}$$

$$\rho\kappa = \frac{1}{c^2}$$

remember: $\nabla^2\underline{E}\left(\underline{r},t\right) - \frac{1}{c^2}\frac{\partial^2\,\underline{E}(\underline{r},t)}{\partial t^2} = 0$

$\nabla^2\underline{B}\left(\underline{r},t\right) - \frac{1}{c^2}\frac{\partial^2\,\underline{B}(\underline{r},t)}{\partial t^2} = 0$

Wave Equation

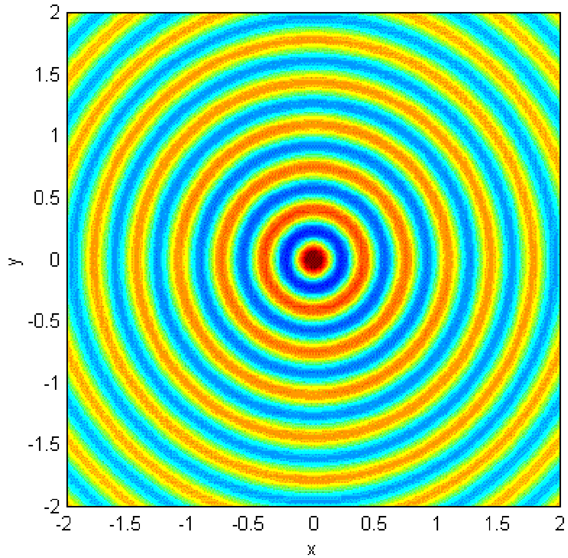
There are two types of sources, which can generate an acoustic field

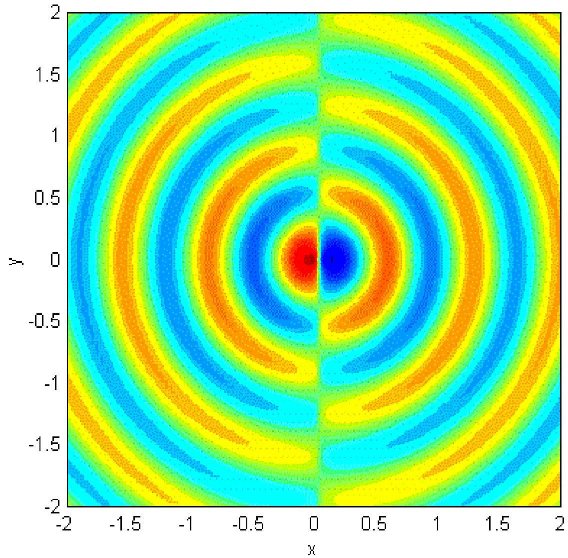
1) A volume source density of injection rate: $q(\underline{r},t)$ [s^{-1}]

2) A volume source density of volume source: $\underline{f}(\underline{r},t)$ [N/m^3]

Hooke's law: $$\frac{\partial p(\underline{r},t)}{\partial t} = -\frac{1}{\kappa}\nabla\cdot\underline{v}(\underline{r},t) + \frac{1}{\kappa}q(\underline{r},t),$$ (equation of deformation)

Newton's law: $$\nabla p(\underline{r},t) = -\rho\frac{\partial\underline{v}(\underline{r},t)}{\partial t} + \underline{f}(\underline{r},t).$$ (equation of motion)





[1] http://en.wikibooks.org/wiki/Aeroacoustics/Acoustic_Sources

Helmholtz Equation

Definition of the temporal Fourier transform of a function $g(\underline{r},t) : \hat{g}(\underline{r},\omega) = \int_{-\infty}^{\infty} g(\underline{r},t)\, e^{-i\omega t} dt$

Fourier transformation of the wave equation $\nabla^2 p(\underline{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\underline{r},t)}{\partial t^2} = 0$

using: $\int_{-\infty}^{\infty} \left(\frac{\partial g(\underline{r},t)}{\partial t}\right) e^{-i\omega t} dt = i\omega\, \hat{g}(\underline{r},\omega)$ yields the Helmholtz equation

$$\boxed{\nabla^2 \hat{p}(\underline{r},\omega) + \frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega) = 0}$$

Spherical Waves

Transforming the Helmholtz equation to polar coordinates
for spherical symmetric solutions yields

$$\nabla^2 \hat{p}(\underline{r},\omega) + \frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega) = -\hat{S}(\underline{r},\omega) \quad \Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\hat{p}(r,\omega)\right) + \frac{\omega^2}{c^2}\hat{p}(r,\omega) = -\delta(\underline{r})\hat{S}(\omega).$$

The most general solution of this equation equals:

$$\hat{p}(r,\omega) = \hat{A}(\omega)\frac{e^{-i\omega r/c_0}}{r} + \hat{B}(\omega)\frac{e^{i\omega r/c_0}}{r}, \quad \text{for } r \neq 0.$$

Why *exp[...]* ?

Why $1/r$?

In practise, with acoustics only the outgoing spherical wave is encountered. In literature, the spherical wave created by a Dirac-delta source is referred to as

Green's function $\hat{G}(\underline{r},\omega)$, or the impulse response of the medium: $\boxed{\hat{G}(\underline{r},\omega) = \frac{e^{-i\omega|\underline{r}|/c_0}}{4\pi|\underline{r}|}}$

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Green's functions

1-D: $\left(\frac{d^2}{dx^2}+k^2\right)\hat{G}(x) = -\delta\left(x-a\right) \quad \rightarrow \quad \hat{G}(x) = \frac{i}{2k}e^{-ikr}$ with $r = |x-a|$

2-D: $\left(\nabla^2+k^2\right)\hat{G}(\vec{x}) = -\delta\left(\vec{x}-\vec{a}\right) \quad \rightarrow \quad \hat{G}(\vec{x}) = -\frac{i}{4}H_0^{(2)}(kr)$ with $r = |\vec{x}-\vec{a}|$

3-D: $\left(\nabla^2+k^2\right)\hat{G}(\vec{x}) = -\delta\left(\vec{x}-\vec{a}\right) \quad \rightarrow \quad \hat{G}(\vec{x}) = \frac{1}{4\pi}\frac{e^{-ikr}}{r}$ with $r = |\vec{x}-\vec{a}|$

Weak form function

The 2-D and 3-D Greens function's are singular for $r=0$. To gives rise to numerical problems when implementing these Green's functions. These problems are solved by using there spherical mean, which is not singular at $r=0$.

2-D: $$\hat{G}(\vec{x}) = -\frac{i}{4} H_0^{(2)}(kr) \quad \rightarrow \quad \begin{cases} \dfrac{i}{2ka} J_1(ka) H_0^{(1)}(kr) & r \geq a \\[2ex] \dfrac{i}{2ka}\left[H_1^{(1)}(ka) + \dfrac{2i}{\pi ka} \right] & r = 0 \end{cases}$$

3-D: $$\hat{G}(\vec{x}) = \frac{1}{4\pi}\frac{e^{-ikr}}{r} \quad \rightarrow \quad \begin{cases} \dfrac{3e^{-ikr}}{4k^3\pi a^3 r}\left[\sin(ka) - ka\cos(ka)\right] & r \geq a \\[2ex] \dfrac{3}{4k^2\pi a^3}\left[(1+ika)e^{-ika} - 1\right] & r = 0 \end{cases}$$

Attenuation

Attenuation is mainly caused by absorption (the transformation of acoustic energy into e.g. heat), geometrical spreading and scattering.

There are various ways to include absorption (the transformation of acoustic energy into e.g. heat) in the field equations. One approach is to use memory or relaxation functions.

In the presence of these memory functions, the acoustic field equations read

Hooke's law: $$\frac{\partial \kappa(t) *_t p(\underline{r},t)}{\partial t} = -\nabla \cdot \underline{v}(\underline{r},t) + q(\underline{r},t)$$ (equation of deformation)

Newton's law: $$\nabla p(\underline{r},t) = -\frac{\partial \rho(t) *_t \underline{v}(\underline{r},t)}{\partial t} + \underline{f}(\underline{r},t)$$ (equation of motion)

with $\kappa(t) = \kappa_0 \delta(t) + \kappa_m(t)$

$\rho(t) = \rho_0 \delta(t) + \rho_m(t).$

The mechanism underlying attenuation is amongst others related to the viscocity of the medium.

Note that in general, attenuation is frequency depenent effect and will give rise to dispersion.

Attenuation and Kramers-Kronig relations

Defining the relaxation functions for the field equations is a difficult process.

Measurements show that for biomedical tissues attenuation may be described via a power law. However, care has to be taken as these relaxation functions should meet the requirements set by nature;

- they should be real valued in the time domain and,
- they should meet the requirements set by causality.

From these requirements, the Kramers-Kronig or generalized dispersion relations are derived:

$$\kappa(\omega) = \kappa_R(\omega) + i\kappa_I(\omega)$$

Complex wave numbers !

with

$$\kappa_R(\omega) = \frac{2}{\pi} p(\omega) \int_0^\infty \frac{\omega' \kappa_I(\omega')}{(\omega')^2 - \omega^2} d\omega' \quad \text{and} \quad \kappa_I(\omega) = -\frac{2}{\pi} p(\omega) \int_0^\infty \frac{\omega' \kappa_R(\omega')}{(\omega')^2 - \omega^2} d\omega'$$

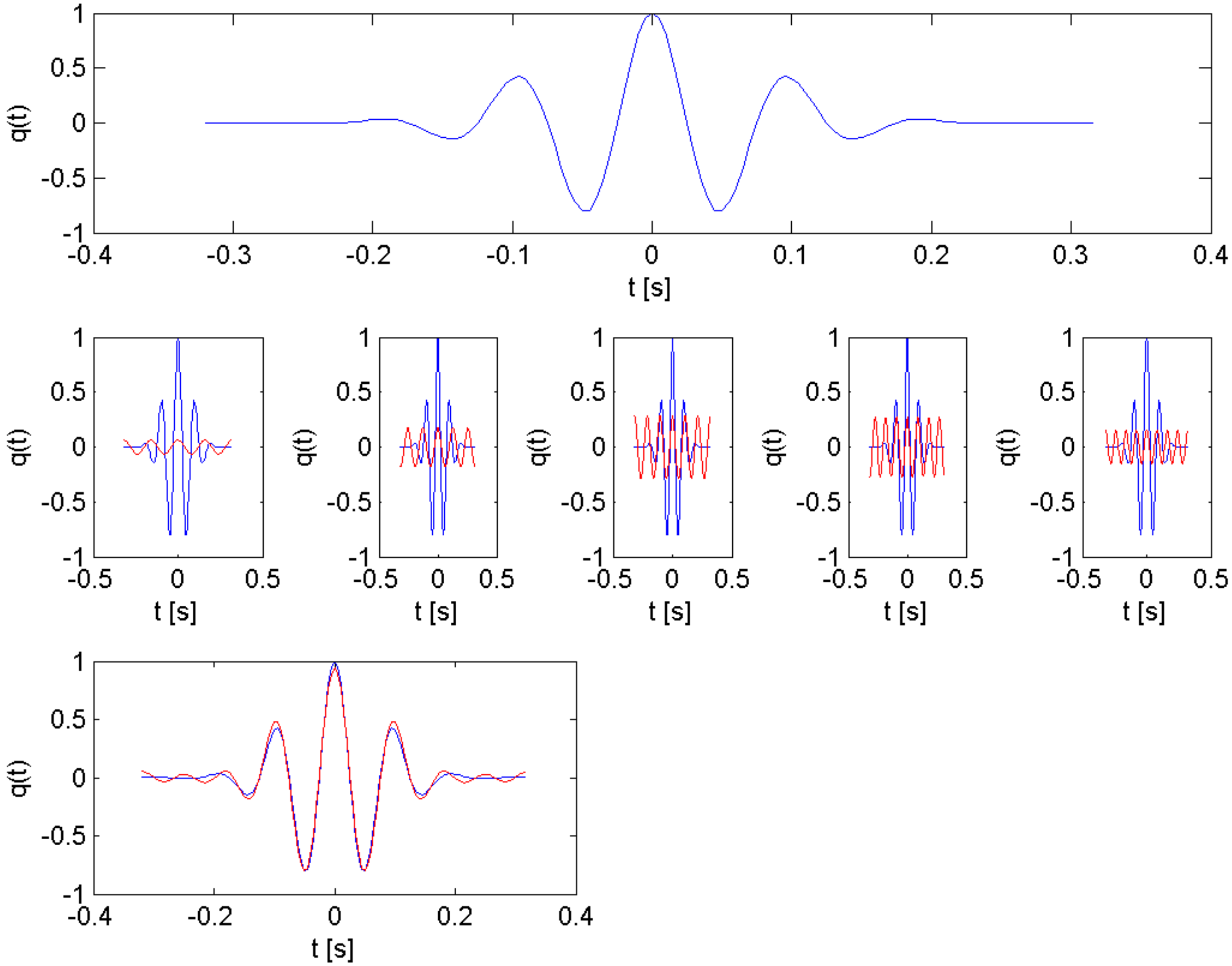
See e.g. M. O'Donnell, E. T. Jaynes, and J. G. Miller, "Kramers-Kronig relationship between ultrasonic attenuation and phase velocity," *J. Acoust. Soc. Amer.* 69(3), 696-701 (1981).

Plane Waves

Any pulse can be described by a combination of sine and cosine functions:

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{+i\omega t}\, d\omega$$



Plane Waves – Acoustic Impedance

The concept of describing a pulse using Fourier series can be extended to 1-D, 2-D and 3-D wave fields, leading to the introduction of plane waves.

General solutions of the Helmholtz equation, $\nabla^2 \hat{p}(\underline{r},\omega) + \frac{\omega^2}{c^2}\hat{p}(\underline{r},\omega) = 0,$

in terms of plane waves equal $\boxed{\hat{p}(\underline{r},\omega) = \sum_{\underline{k}} F_{\underline{k}}(\omega) e^{-i\underline{k}\cdot\underline{r}}}$, with wave vector $\underline{k}$ of length $|\underline{k}| = \frac{\omega}{c}$.

Calculating the velocity field that goes along with these pressure plane waves yields

$$\left.\begin{array}{l} -\nabla \hat{p}(\underline{r},\omega) = i\omega\rho_0 \hat{\underline{v}}(\underline{r},\omega) \\ \hat{p}(\underline{r},\omega) = \sum_{\underline{k}} F_{\underline{k}}(\omega) e^{-i\underline{k}\cdot\underline{r}} \end{array}\right\} \Rightarrow \sum_{\underline{k}} i\underline{k}\, F_{\underline{k}}(\omega) e^{-i\underline{k}\cdot\underline{r}} = i\omega\rho \hat{\underline{v}}(\underline{r},\omega) \Rightarrow \hat{\underline{v}}(\underline{r},\omega) = \frac{1}{\rho c}\sum_{\underline{k}} \frac{\underline{k}}{|\underline{k}|} F_{\underline{k}}(\omega) e^{-i\underline{k}\cdot\underline{r}}$$

1-D : $\Rightarrow$ $\boxed{\hat{p}(x,\omega) = \rho c\, \hat{v}(x,\omega) = Z\, \hat{v}(x,\omega)}$, where Z is refered to as the acoustic impedance.

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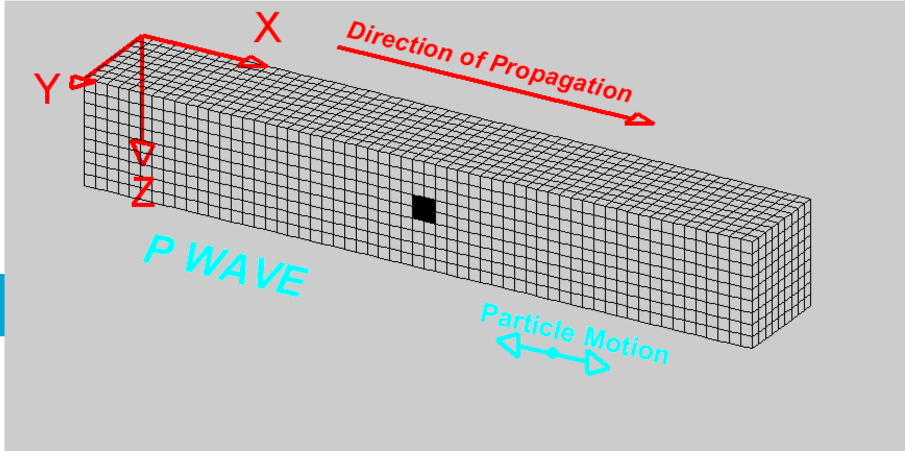
Longitudinal wave propagation

If we take a pressure wave which travels only in the x-direction, we obtain the following expression:

$$\hat{p}\left(\underline{r},\omega\right)=\sum_{\underline{k}}F_{\underline{k}}(\omega)e^{-i\underline{k}\cdot\underline{r}}=F(\omega)e^{-i\frac{\omega}{c}x}$$

For this expression, the velocity wave equals $\hat{\underline{v}}\left(\underline{r},\omega\right)=\frac{1}{\rho c}\sum_{\underline{k}}\frac{\underline{k}}{\left|\underline{k}\right|}F_{\underline{k}}(\omega)e^{-i\underline{k}\cdot\underline{r}}=\frac{1}{\rho c}F(\omega)e^{-i\frac{\omega}{c}x}=\hat{p}\left(\underline{r},\omega\right).$

Generalising this to three dimensions shows that the particle movement is in the direction of propagation, leading to longitudinal waves.



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Impedance Single Outgoing Spherical Wave

Outgoing spherical wave:

$$p(r,t) = \frac{f(t - r/c)}{r}$$

In the frequency domain:

$$\hat{p}(r,\omega) = F(\omega)\frac{e^{-i\omega r/c}}{r}$$

Calculate particle velocity from Newton's law:

$$\underline{\hat{v}}(r,\omega) = -\frac{1}{i\omega\rho}\nabla\hat{p}(r,\omega) \quad , \quad \nabla\hat{p}(r,\omega) = \frac{\partial\hat{p}}{\partial r}\underline{\hat{r}}$$

$$\underline{\hat{v}}(r,\omega) = \frac{1}{\rho c}\left(1 + \frac{1}{i\omega r/c}\right)F(\omega)\frac{e^{-i\omega r/c}}{r}\underline{\hat{r}} = \frac{1}{\rho c}\left(1 + \frac{1}{i\omega r/c}\right)\hat{p}(r,\omega)\underline{\hat{r}}$$

Impedance: $$\boxed{Z(r,\omega) = \frac{\hat{p}(r,\omega)}{\hat{v}(r,\omega)} = \frac{\rho c}{1 + \dfrac{1}{i\omega r/c}}}$$, $\left(\underline{\hat{v}}(r,\omega) = \hat{v}(r,\omega)\underline{\hat{r}} \right)$

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Near and Far Field approximations for Spherical Waves

The impedance Z equals $Z(r,\omega) = \dfrac{\hat{p}(r,\omega)}{\hat{v}(r,\omega)} = \dfrac{\rho c}{1 + \dfrac{1}{i\omega r/c}} = \dfrac{\rho c\left(1 + \dfrac{ic}{\omega r}\right)}{1 + \dfrac{c^2}{\omega^2 r^2}}$

In the far field ($\omega r/c >> 1$) we have:

$$\hat{p}(r,\omega) \approx \rho c\, \hat{v}(r,\omega)$$

The impedance becomes real valued and hence *resistive.* In the far field spherical waves behave like plane waves.

In the near field ($\omega r/c << 1$) we have:

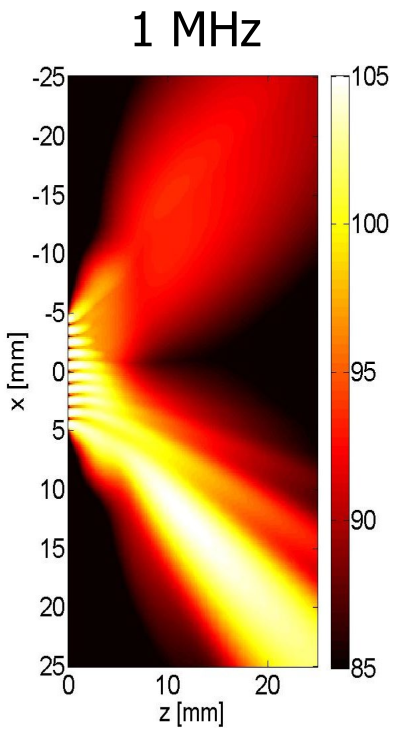
$$\hat{p}(r,\omega) \approx i\omega\rho r\, \hat{v}(r,\omega)$$

This gives a predominantly imaginary impedance or a *reactive* impedance.

Now the pressure and the velocity wave field are out of phase!

Nonlinear Acoustics

Arrays are used to steer the beam into a certain direction.
Due to the finite size of the elements and the spacing in between elements, side and grating lobes will occur, leading to a blurring of the image.



Nonlinear Acoustics

To derive the linear wave equation, three approximations were made

- the volume density of mass was taken to be constant (pressure independent):

$$\rho_0(p) = \rho_0(p_0) + \cancel{(p - p_0)\left.\left(\frac{\partial \rho_0(p)}{\partial p}\right)\right|_{p=p_0}},$$

- the bulk modulus (or compressibility) was taken to be constant (pressure independent):

$$\kappa_0(p) = \kappa_0(p_0) + \cancel{(p - p_0)\left.\left(\frac{\partial \kappa_0(p)}{\partial p}\right)\right|_{p=p_0}},$$

- the convection term of the material derivative was taken to be zero:

$$\frac{dp(\underline{r},t)}{dt} \equiv \frac{\partial p(\underline{r},t)}{\partial t} + \cancel{\underline{v} \cdot \nabla p(\underline{r},t)}$$

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Nonlinear Acoustics

For linear acoustics, experiments show that the previous assumptions are valid.

However, for high amplitude pressure fields, this approximation is no longer valid, moreover the volume density of mass $\rho(p)$ may be approximated by

$$\left.\begin{array}{l} \rho(p) = \rho_0 + (p - p_0)\left.\dfrac{\partial \rho}{\partial p}\right|_{p-p_0} \\[2ex] p_0\left(\Delta V\right)^{\gamma} = const \;\Rightarrow\; p_0\left(\rho_0\right)^{-\gamma} = p\left(\rho\right)^{-\gamma} \;\Rightarrow\; \dfrac{\partial \rho}{\partial p} = \dfrac{\rho}{\gamma p} \\[2ex] \gamma p_0 = \dfrac{1}{\kappa_0} \end{array}\right\} \Rightarrow \boxed{\rho(p) = \rho_0\left[1 + \kappa_0(p - p_0)\right]}$$

A similar expression may be obtained for the compressibilitiy $\kappa(p)$

$$\boxed{\kappa(p) = \kappa_0\left[1 + \kappa_0(1 - 2\beta)(p - p_0)\right]}$$

with β the coëfficient of non-linearity.

Speed of Sound

Combination of the second order approximations

$$\frac{1}{c(p)^2} = \kappa(p)\rho(p)$$

$$= \kappa_0\rho_0\left[1+\kappa_0 p)\right]\left[1+\kappa_0(1-2\beta)p\right]$$

$$= \kappa_0\rho_0\left[1+2\kappa_0(1-\beta)p+\kappa_0^2(1-2\beta)p^2\right].$$

Typical values for β vary from 3.6 (water) to 10 (methane).

Hence, for

- increasing pressure we observe an increase in speed of sound c,
- decreasing pressure we observe an decrease in speed of sound c,

This will lead to a change of the shape of the waveform of the wavefield.

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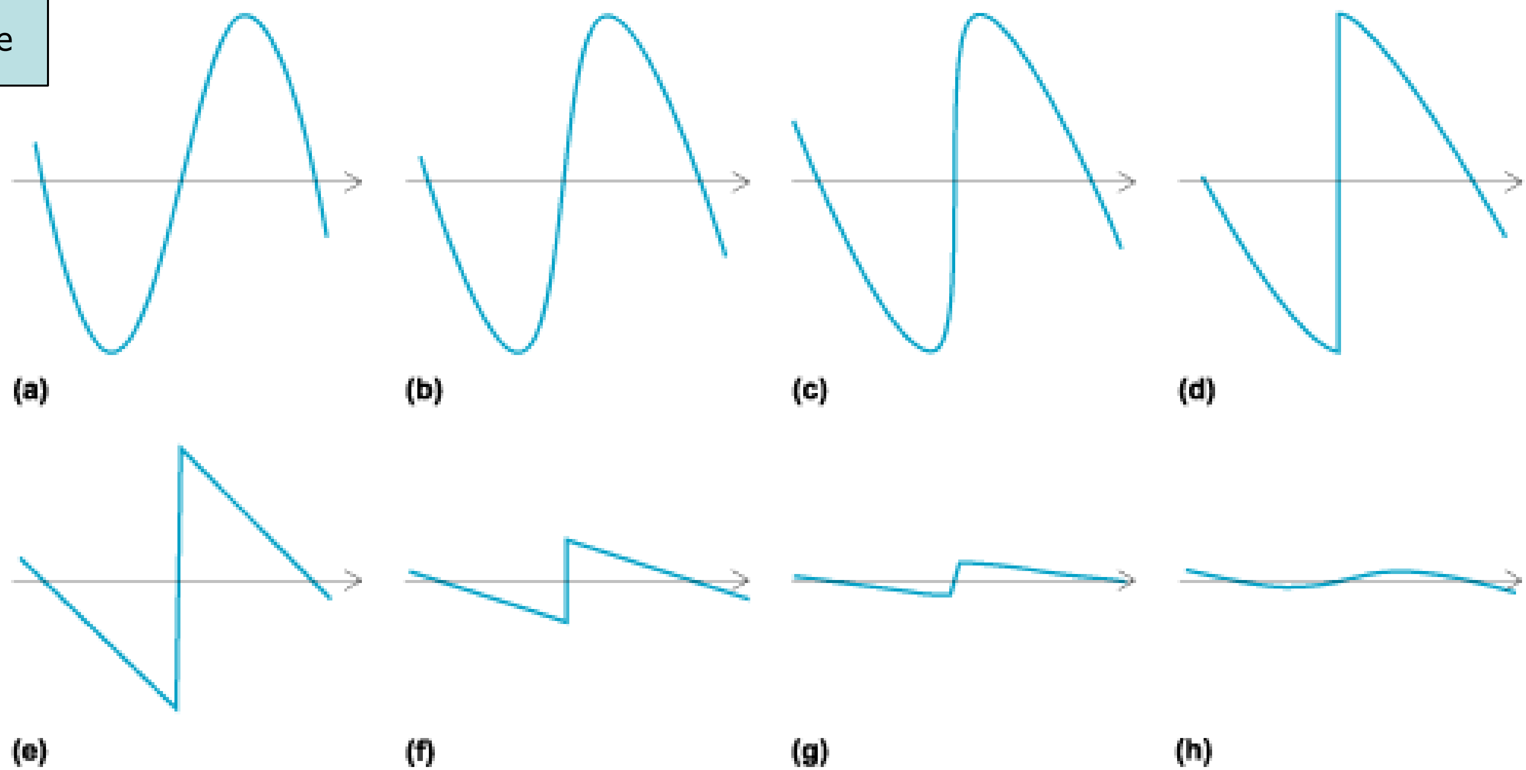
Nonlinear Propagation

Propagation history of an intense acoustic waveform that is sinusoidal at the source.

(a) $x = 0$: source waveform, (b) distortion becoming noticeable,
(c) $x = x_s$: shock formation, (d) $x = (\pi/2)\, x_s$ maximum shock amplitude,
(e) $x = 3\, x_s$ full sawtooth shape, (f) decaying sawtooth,
(g) shock beginning to disperse, (h) old age.

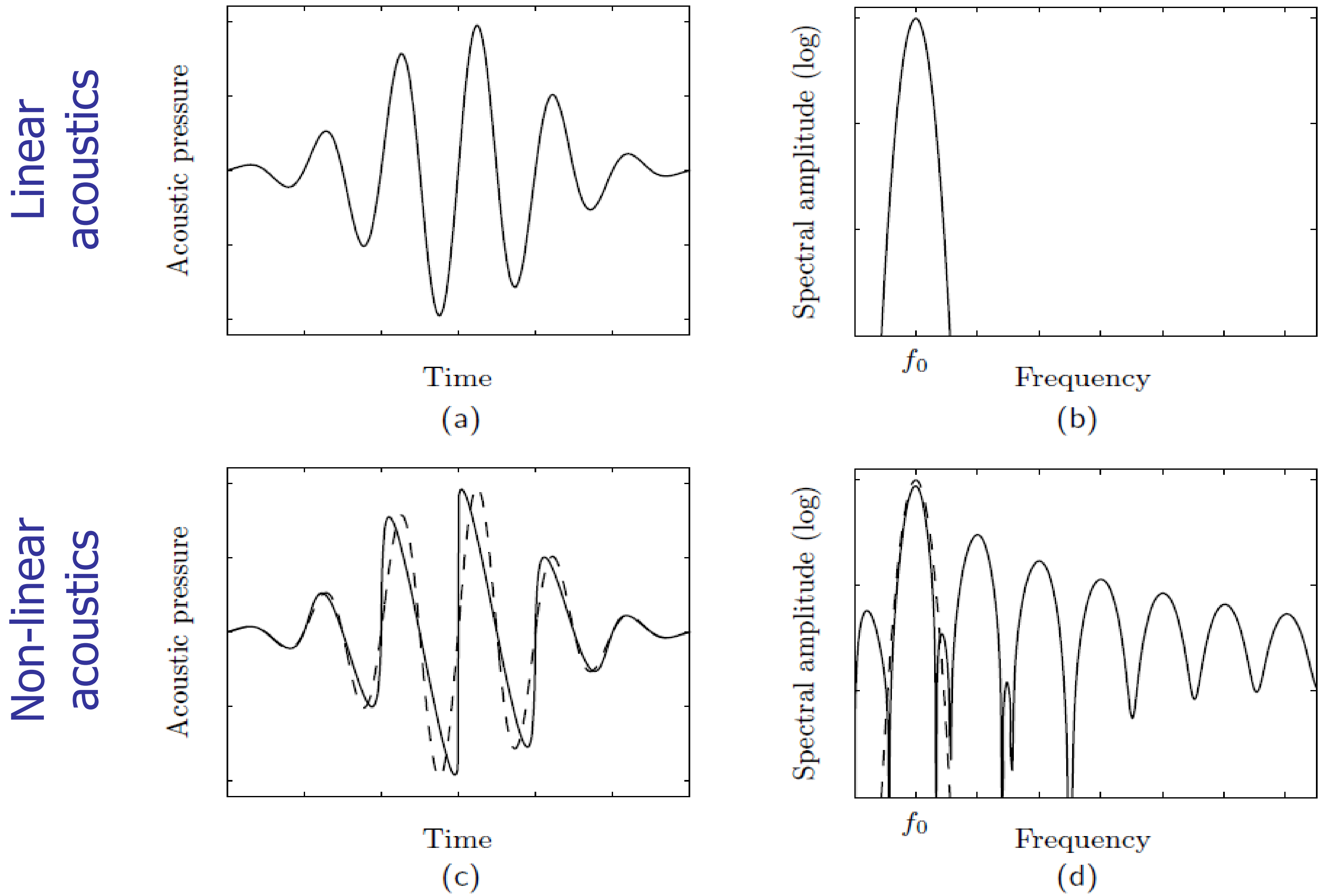
(After J. A. Shooter et al., Acoustic saturation of spherical waves in water, J. Acous. Soc. Amer., 55:54–62, 1974)

Display oscilloscope



Linear and Nonlinear Propagation

The nonlinear propagation leads to a steepening of the wave form. In the frequency domain this corresponds to the formation of higher harmonic components.



Linear and Nonlinear Propagation

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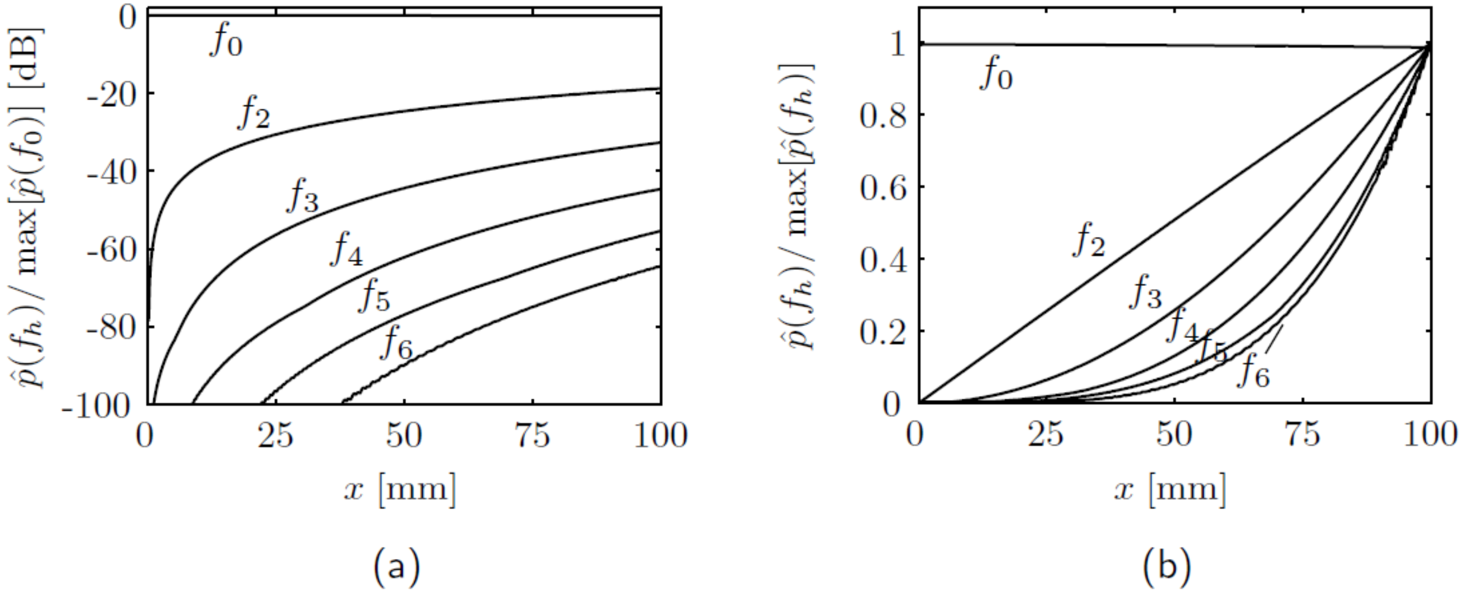


Nonlinear Wave Equation

Combination of the second order approximations

$$\rho(p) = \rho_0\left[1+\kappa_0(p-p_0)\right],$$
$$\kappa(p) = \kappa_0\left[1+\kappa_0(1-2\beta)(p-p_0)\right]$$

with Hooke's law $\nabla \underline{v} + \rho D_t p = 0,$ and Newtons law $\nabla p + \kappa D_t \underline{v} = 0,$

leads to the following set of equations

$$\nabla \underline{v} + \left(\rho_0\left[1+\kappa_0(p-p_0)\right]\right)\left(\partial_t + \underline{v}\cdot\nabla\right) p = 0$$
$$\nabla p + \left(\kappa_0\left[1+\kappa_0(1-2\beta)(p-p_0)\right]\right)\left(\partial_t + \underline{v}\cdot\nabla\right) \underline{v} = 0.$$

Combining the above set of equations and neglecting terms of third order and higher yields the <u>second-order nonlinear wave equation</u>

$$\boxed{\nabla^2 p - \frac{1}{c^2}\partial_t^2 p = -\frac{\beta}{\rho_0 c_0^4}\partial_t^2 p^2}$$

Nonlinear Wave Equation

Various methods exist to model nonlinear propagation. If the nonlinearity is weak, the additional term may be considered as a contrast source. Next, a solution for the Westervelt equation, which equals

$$\nabla^2 P(\underline{r}) - \frac{1}{c^2}\partial_t^2 P(\underline{r}) = -S^{prime}(\underline{r}) - \frac{\beta}{\rho_0 c_0}\partial_t^2 P^2(\underline{r}),$$

may be obtained by recasting the differential equation into an integral equation, viz.

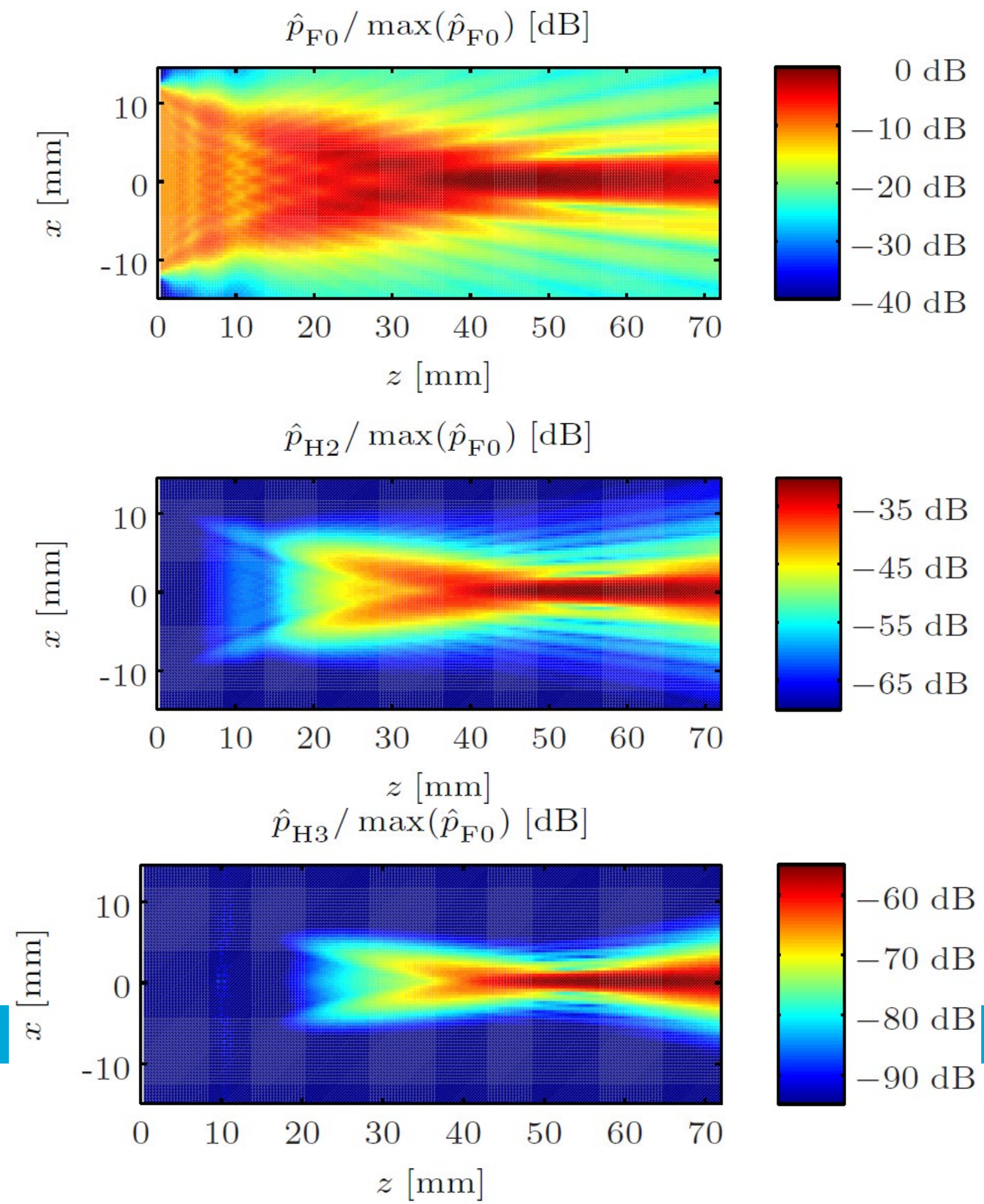
$$P(\underline{r}) = P^{inc}(\underline{r}) - \int_{\underline{r}\in D} G(\underline{r} - \underline{r}') \frac{\beta}{\rho_0 c_0^4}\omega^2 \left(P(\underline{r}) *_\omega P(\underline{r}) \right) dV.$$

If the nonlinearity is weak, a Neumann scheme is sufficient to solve the integral equation. Note that with each iteration step, one additional harmonic is formed.

Harmonic Imaging

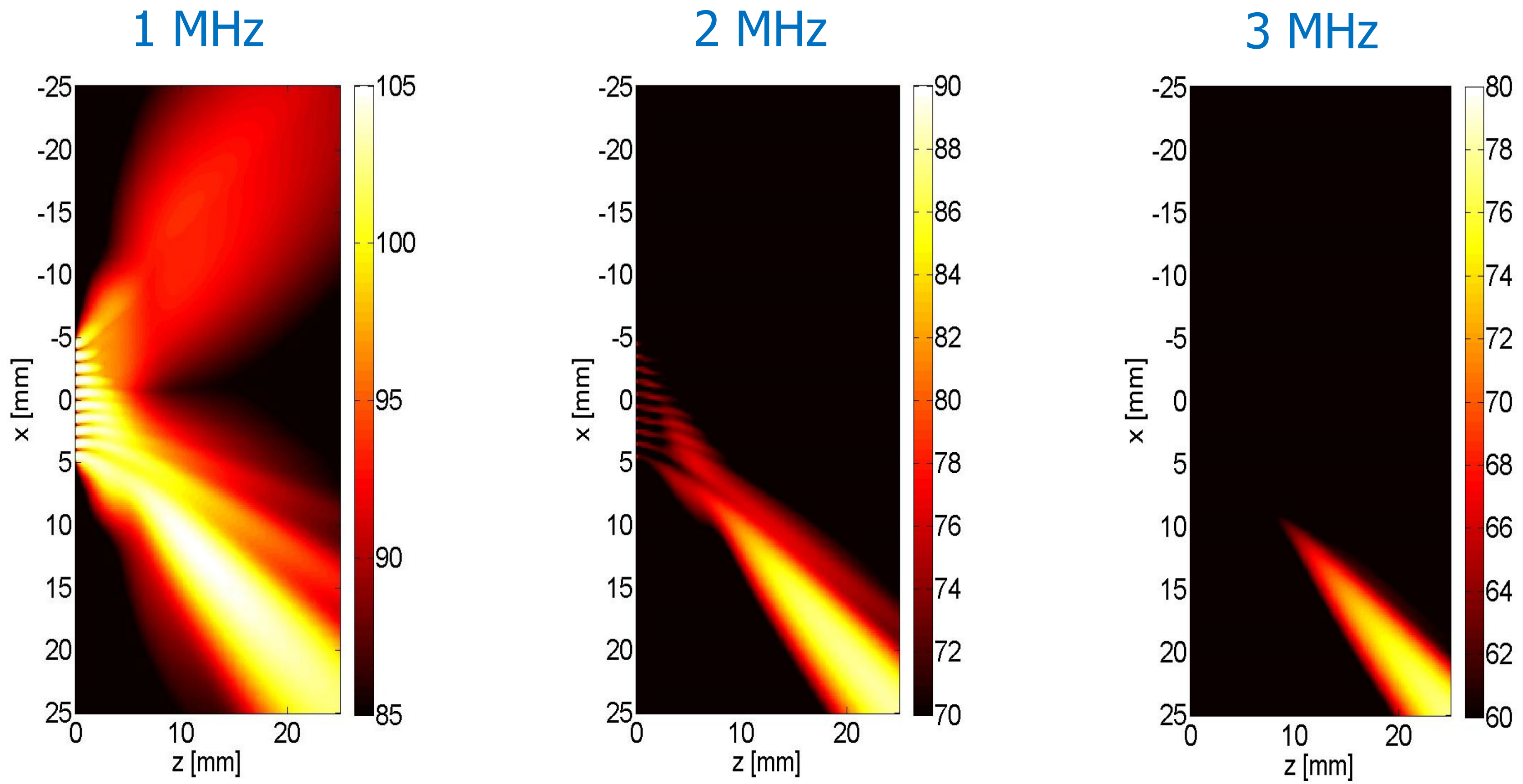
Cardiac Imaging is a well known application for harmonic imaging, as the harmonic components are formed behind the ribs:

- no reflections from the ribs;
- a narrow beam profile.



Harmonic Imaging

The nonlinear propagation can be used to suppress side lobes which are mainly present in fundamental beam.



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Hyperthermia with High Intensity Focussed Ultrasound

- HIFU
  - Hyperthermia T = ± 45 °C
  - Ablation T > 60 °C
- Porcine liver

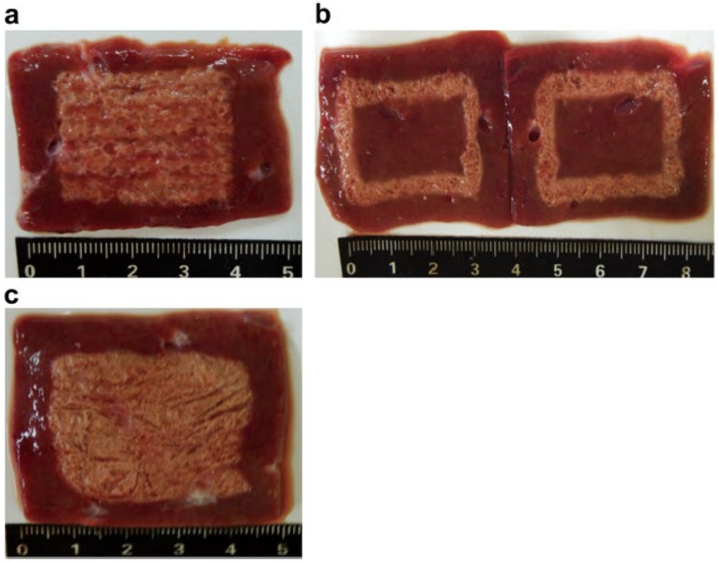


Fig. 6. Gross assessment after peripheral scanning with HIFU. After HIFU ablation according to the peripheral scanning protocol with dual-vessel perfusion, target tissues were cut into 3-mm sections along the height of each sample (*i.e.*, along the *z*-axis). (a) Top surface (*i.e.*, 40-mm depth): with identical generator power (350 W), scanning velocity (4 mm/s) and interval between two adjacent lines (4 mm). (b) Intermediate sections (*i.e.*, 34- and 28-mm depths): at 34-mm depth (left), generator power = 350 W, scanning velocity = 5 mm/s; at 28-mm depth (right), generator power = 350 W, scanning velocity = 6 mm/s. The straight lines scanned only along the periphery of the target region. (c) Undersurface (*i.e.*, 22-mm depth): with identical generator power (300 W), scanning velocity (6 mm/s) and interval between two adjacent lines (4 mm).

Tissue Ablation Accelerated by Peripheral Scanning Mode with High-Intensity Focused Ultrasound: A Study on Isolated Porcine Liver Perfusion Bu, Rui et al. Ultrasound in Medicine and Biology , Volume 39 , Issue 8 , 1410 - 1419

Acoustic Streaming (movies)

- First observed by Faraday in air in 1831.

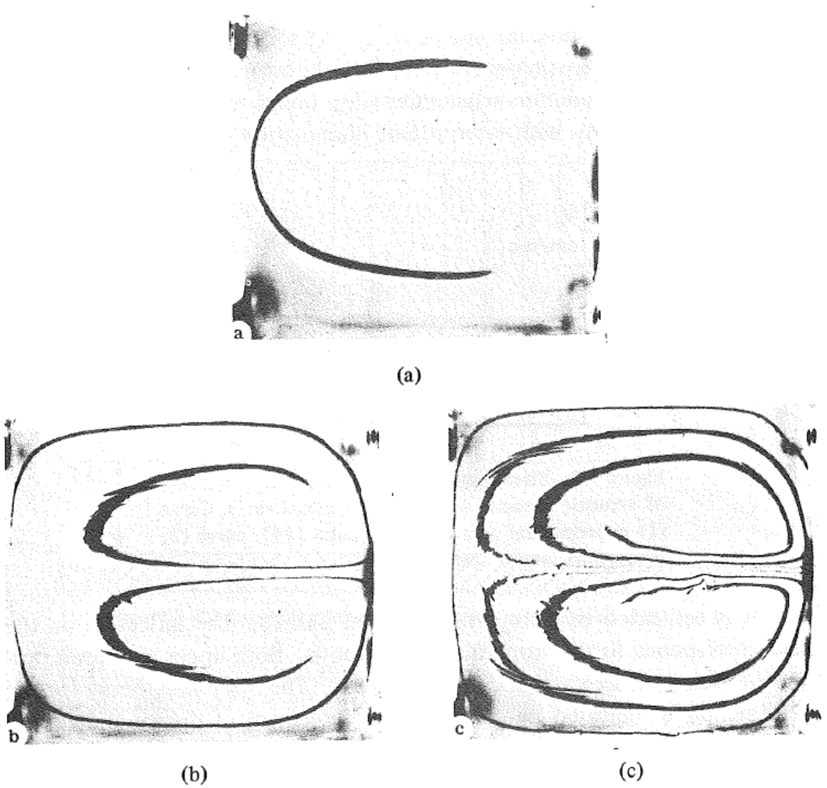


Figure 7-5.–Motion of a colored film of water at an interface between glycerine and vaseline for increasing time after start of experiment. Source is at the left (from Zarembo and Skhlovskaya-Kordi [12]).

R. T. Beyer, "Nonlinear Acoustics" (1997)

https://youtu.be/XkoHYD3CDo0

Christiaan Huygens

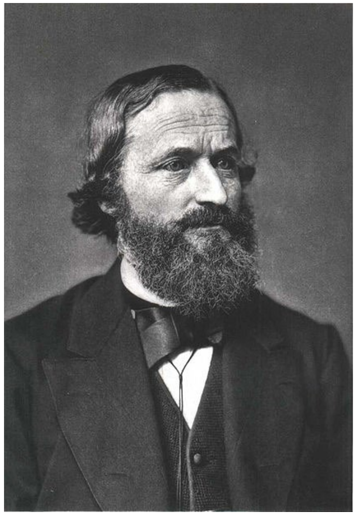
14 April 1629 (The Hague)
8 July 1695 (The Hague)

Augustin Jean Fresnel

10 May 1788 (Broglie)
14 July 1827 (Ville-d'Avray)

George Green

14 July 1793 (Sneinton)
31 May 1841 (Sneinton)

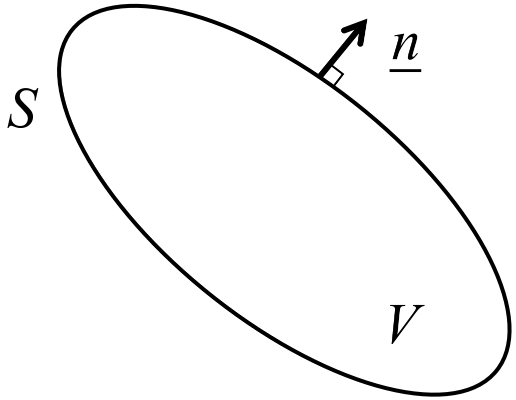


Gustav Robert Kirchhoff

12 March 1824
(Koningsbergen)
17 Oct. 1887 (Berlin)

[1] http://nl.wikipedia.org/wiki/Christiaan_Huygens
[2] http://nl.wikipedia.org/wiki/Augustin_Jean_Fresnel
[3] http://www.nottingham.ac.uk/physics/about/history/george-green.aspx
[4] http://nl.wikipedia.org/wiki/Gustav_Robert_Kirchhoff

Green's Theorem



Consider Gauss' Theorem for any arbitrary vector field $\underline{a}(\underline{r})$ and arbitrary volume V, bounded by surface S :

$$\int_V \nabla \cdot \underline{a}(\underline{r})\, dV = \oint_S \underline{a}(\underline{r}) \cdot \underline{n}\, dS$$

For two functions f and g that are twice differentiable we can write:

$$\underline{a}(\underline{r}) = f(\underline{r}) \nabla g(\underline{r}) \qquad \Rightarrow \qquad \nabla \cdot \underline{a} = f \nabla^2 g + \nabla f \cdot \nabla g$$

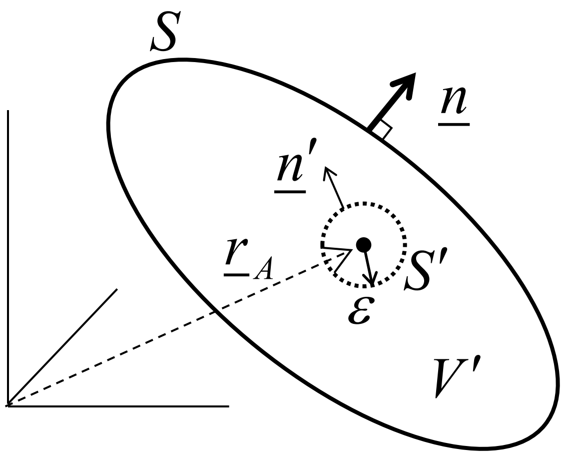
and: $$\underline{a}'(\underline{r}) = g(\underline{r}) \nabla f(\underline{r}) \qquad \Rightarrow \qquad \nabla \cdot \underline{a}' = g \nabla^2 f + \nabla g \cdot \nabla f$$

$$\Rightarrow \qquad \nabla \cdot (\underline{a} - \underline{a}') = f \nabla^2 g - g \nabla^2 f$$

$$\Rightarrow \qquad \boxed{\int_V \left(f \nabla^2 g - g \nabla^2 f \right) dV = \oint_S \left(f \nabla g - g \nabla f \right) \cdot \underline{n}\, dS}$$

This is *Green's Second Identity.*

Kirchhoff Integral



For f substitute $\hat{p}(\underline{r})$, which is a solution of the Helmholtz equation:

$$\nabla^2 \hat{p} + \frac{\omega^2}{c^2}\hat{p} = 0$$

everywhere in the volume V.

For g substitute $\hat{G}(\underline{r}) = \frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|}$, which is a solution of:

$$\nabla^2 \hat{G}(\underline{r}) + \frac{\omega^2}{c^2}\hat{G}(\underline{r}) = -\delta(\underline{r}-\underline{r}_A)$$

in the volume V', which is inside S, *but outside* the surface S' around the point-source at $\underline{r}_A$ inside V.

$$\Rightarrow \qquad \int_{V'}\left(\hat{p}\nabla^2\hat{G} - \hat{G}\nabla^2\hat{p}\right)dV' = \int_{V'}\left(-\hat{p}\frac{\omega^2}{c^2}\hat{G} + \hat{G}\frac{\omega^2}{c^2}\hat{p}\right)dV' \equiv 0$$

$$\Rightarrow \qquad \oiint_{S}\left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}\,dS - \oiint_{S'}\left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}'\,dS' = 0$$

Kirchhoff Integral

For the integral over S' we can write:

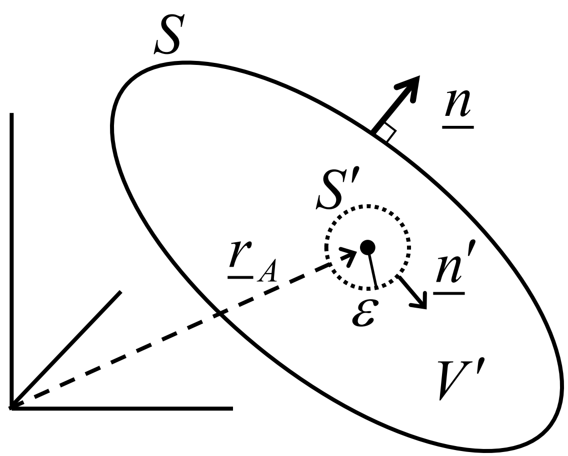
$$\lim_{\varepsilon\to 0} \oiint_{S'} \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p} \right)\cdot \underline{n}'\, dS'$$

$$= \lim_{\varepsilon\to 0} \int_0^{\pi} \int_0^{2\pi} \left[\hat{p}\left(\underline{r}_A + \cancel{\varepsilon\underline{n}'}\right) \frac{\partial}{\partial\varepsilon}\left(\frac{e^{-i\omega\varepsilon/c}}{4\pi\varepsilon} \right) - \cancel{\frac{e^{-i\omega\varepsilon/c}}{4\pi\varepsilon}\left(\nabla\hat{p}\cdot\underline{n}'\right)} \right] \varepsilon^2 \sin\vartheta\, d\varphi\, d\vartheta$$

$$= \lim_{\varepsilon\to 0} \int_0^{\pi} \int_0^{2\pi} \left[\hat{p}\left(\underline{r}_A\right) \left(-\frac{1}{\varepsilon^2} - \cancel{\frac{i\omega}{c\varepsilon}} \right) e^{-i\omega\varepsilon/c} \frac{1}{4\pi} \right] \varepsilon^2 \sin\vartheta\, d\varphi\, d\vartheta$$

$$= -\,\hat{p}\left(\underline{r}_A\right)$$

$$\Rightarrow \qquad \boxed{\hat{p}\left(\underline{r}_A\right) = -\oiint_S \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p} \right)\cdot \underline{n}\, dS}$$



This is the *Kirchhoff integral.*

$$\hat{G}(\underline{r},\underline{r}_A,\omega) = \frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi\left|\underline{r}-\underline{r}_A\right|}$$ is called the *Green's function*.

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Kirchhoff Integral

$$\hat{p}\left(\underline{r}_A\right) = -\oint_S \left(\hat{p}\nabla\hat{G} - \hat{G}\nabla\hat{p}\right)\cdot\underline{n}\, dS \qquad \text{(point } A \text{ inside } S \text{)}$$

- The *Green's function* $\hat{G}(\underline{r},\omega)$ is the field of a point source located at $\underline{r}_A$ with delta-pulse wave-form.

- The field $\hat{G}$ does not co-exist with $\hat{p}$ in the same experiment. It is only introduced mathematically through Green's theorem.

- The Kirchhoff integral allows us to calculate the wave field $\hat{p}\left(\underline{r},\omega\right)$ at position $\underline{r}_A$, from recordings of $\hat{p}$ and $\left(\nabla\hat{p}\right)_n$ along any closed surface S around A.

- Application of the Kirchhoff integral can be cumbersome because:
  - We need recordings for both $\hat{p}$ and $\left(\nabla\hat{p}\right)_n$.
  - We need recordings along a closed surface.

- Under some limiting conditions there is a trick to be applied that circumvents both problems simultaneously.

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Rayleigh Integral

In the Kirchhoff integral:

$$\hat{p}\left(\underline{r}_A,\omega\right) = -\oint_S \left(\hat{p}\,\nabla\hat{G} - \hat{G}\,\nabla\hat{p}\right)\cdot\underline{n}\;dS$$

there is a degree of freedom, since for any function $\hat{\Gamma}\left(\underline{r},\omega\right)$ that satisfies:

$$\nabla^2\hat{\Gamma} + \frac{\omega^2}{c^2}\hat{\Gamma} = 0$$

everywhere inside S, we can write:

$$\hat{p}\left(\underline{r}_A\right) = -\oint_S \left[\hat{p}\,\nabla\left(\hat{G}+\hat{\Gamma}\right) - \left(\hat{G}+\hat{\Gamma}\right)\nabla\hat{p}\right]\cdot\underline{n}\;dS$$

Obviously this is the case because:

$$\oint_S \left[\hat{p}\,\nabla\hat{\Gamma} - \hat{\Gamma}\,\nabla\hat{p}\right]\cdot\underline{n}\;dS = \int_V \left[\hat{p}\nabla^2\hat{\Gamma} - \hat{\Gamma}\nabla^2\hat{p}\right]dV = 0$$

Rayleigh Integral

We want to use the function Γ in the Kirchhoff integral:

$$\hat{p}\left(\underline{r}_A,\omega\right)=-\oiint_S\left[\cancel{\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right)}-\left(\hat{G}+\hat{\Gamma}\right)\nabla\hat{p}\right]\cdot\underline{n}\,dS$$

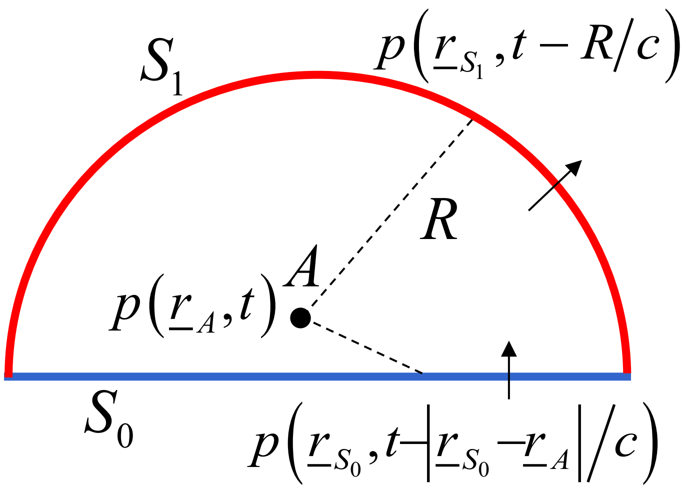
in such a way that either the term with $\hat{p}$ or the term with $\nabla\hat{p}$ vanishes over the relevant part of S.

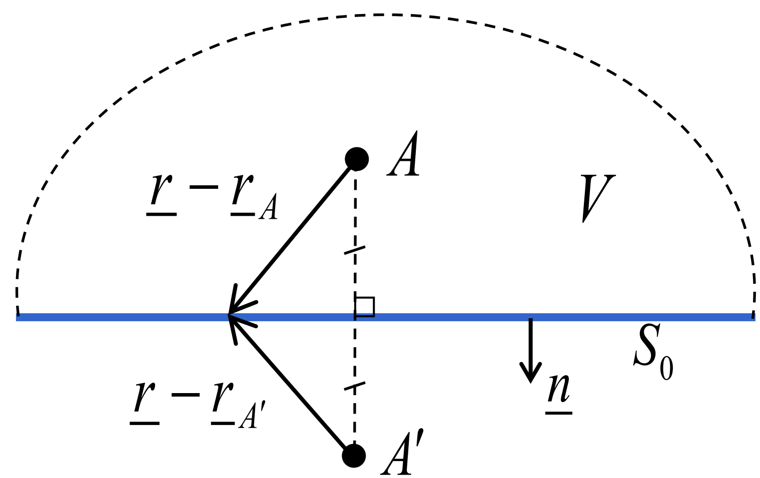
What the relevant part of S is depends on where the sources are that generated the wave field $\hat{p}$ and whether we want to predict forward or backward in time.

If there are sources in all directions from the point A , the whole closed surface S is relevant and there exists no suitable choice for $\hat{\Gamma}$ that simplifies the Kirchhoff integral.

Rayleigh Integral

$$\hat{p}(\underline{r}_A,\omega) = -\oint_S \left[\hat{p}\nabla(\hat{G}+\hat{\Gamma}) - (\hat{G}+\hat{\Gamma})\nabla\hat{p}\right]\cdot\underline{n}\,dS$$





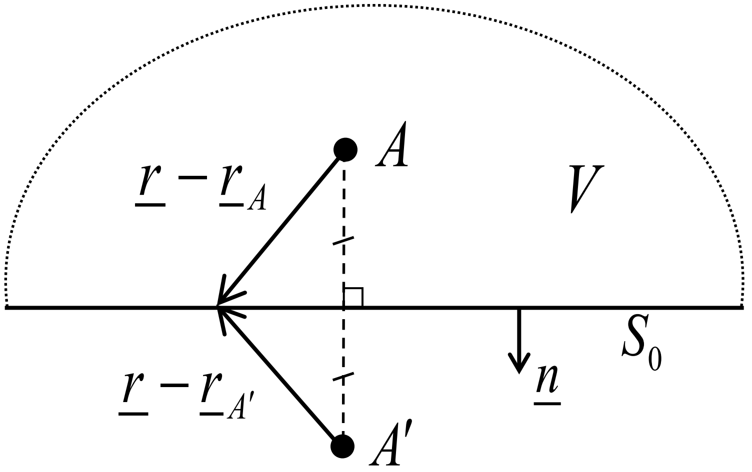
$$\hat{p}(\underline{r}_A,\omega) = \oint_S \left[(\hat{G}+\hat{\Gamma})\nabla\hat{p}\right]\cdot\underline{n}\,dS \quad \leftarrow$$

$\nabla\left[\hat{G}(\underline{r})+\hat{\Gamma}(\underline{r})\right] = 0$ for all $\underline{r}\in S_0$ (Rayleigh I)

$$\hat{p}(\underline{r}_A,\omega) = -\oint_S \hat{p}\nabla(\hat{G}+\hat{\Gamma})\cdot\underline{n}\,dS \quad \leftarrow$$

$\hat{G}(\underline{r})+\hat{\Gamma}(\underline{r}) = 0$ for all $\underline{r}\in S_0$ (Rayleigh II)

Rayleigh II Integral



We have established that in the case that all the sources of the field $\hat{p}$ are below the plane S_0 , we only need to integrate over S_0 .

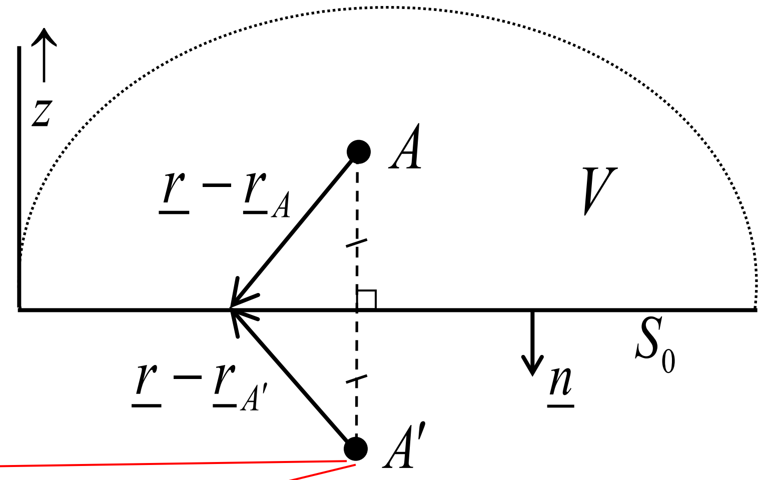
We now try to find a function $\hat{\Gamma}(\underline{r},\omega)$

that makes $\boxed{\hat{G}+\hat{\Gamma}=0 \text{ everywhere on } S_0}$.

Recalling that $\hat{G}$ is the wave field of a point source in point A , we can create a wave field $\hat{\Gamma}$ by putting a point-source with a negative source strength in the mirror point of A : A' .

This is legitimate because then the field $\hat{\Gamma}$ is not created by sources inside V and so satisfies the equation $\nabla^2\hat{\Gamma}+\left(\omega^2/c^2\right)\hat{\Gamma}=0$ everywhere inside V.

Rayleigh II Integral



If $\hat{G}+\hat{\Gamma}=0\Big|_{\vec{r}\in S_0}$, the Kirchhoff integral reduces to:

$$\hat{p}\left(\underline{r}_A\right)=-\int_{S_0}\left[\hat{p}\,\nabla\left(\hat{G}+\hat{\Gamma}\right)\right]\cdot\underline{n}\,dS_0$$

with:

$$\hat{G}(\underline{r})=\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|} \quad \text{and} \quad \hat{\Gamma}(\underline{r})=-\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi|\underline{r}-\underline{r}_{A'}|}$$

where:

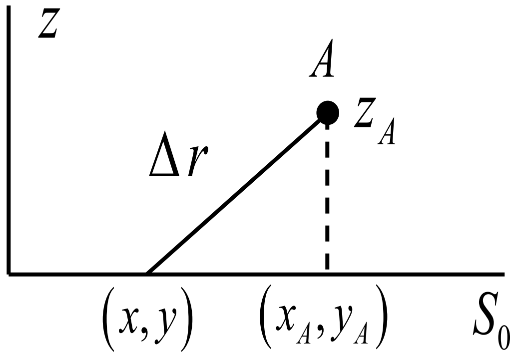
$$|\underline{r}-\underline{r}_A|=\sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+\left(z-z_A\right)^2}$$

$$|\underline{r}-\underline{r}_{A'}|=\sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+\left(z+z_A\right)^2}$$

On S_0 we have:

$$z=0 \quad , \quad |\underline{r}-\underline{r}_A|=|\underline{r}-\underline{r}_{A'}| \quad \text{and} \quad \nabla\left(\hat{G}+\hat{\Gamma}\right)\cdot\underline{n}=-\left[\frac{\partial}{\partial z}\left(\hat{G}+\hat{\Gamma}\right)\right]_{z=0}$$

Rayleigh II Integral



$$\frac{\partial}{\partial z}\left(\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|}\right)=-\frac{z-z_A}{|\underline{r}-\underline{r}_A|^3}\left(1+i\omega\frac{|\underline{r}-\underline{r}_A|}{c}\right)\frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi}$$

$$\frac{\partial}{\partial z}\left(-\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi|\underline{r}-\underline{r}_{A'}|}\right)=\frac{z+z_A}{|\underline{r}-\underline{r}_{A'}|^3}\left(1+i\omega\frac{|\underline{r}-\underline{r}_{A'}|}{c}\right)\frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi}$$

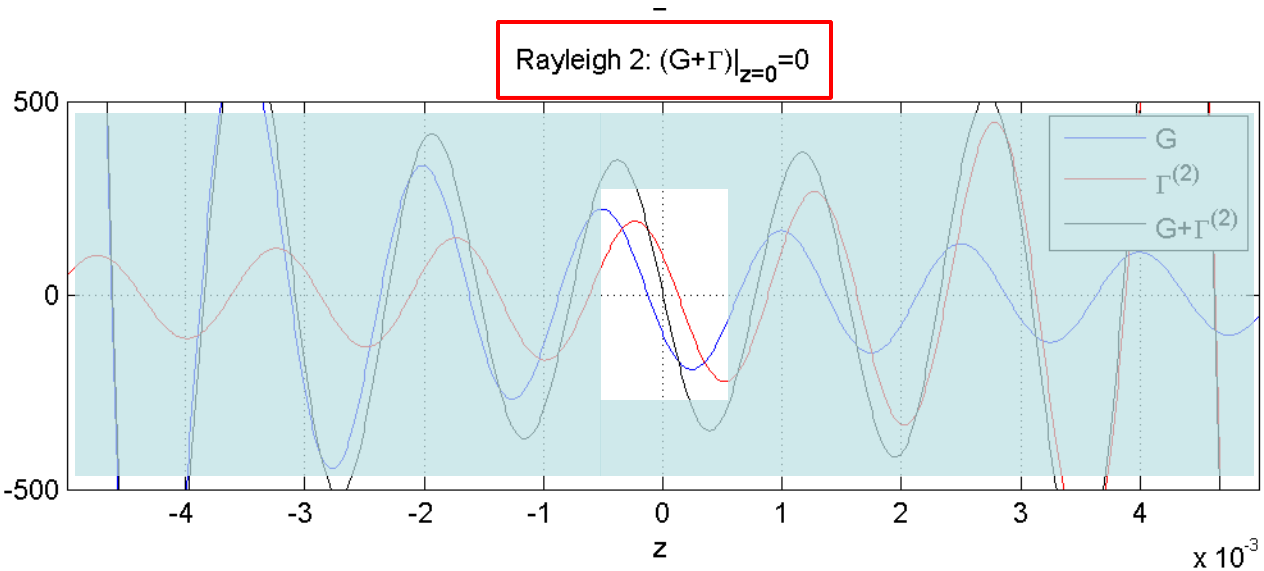
$$\left[\frac{\partial}{\partial z}\left(\hat{G}+\hat{\Gamma}\right)\right]_{z=0}=\frac{z_A\left(1+i\omega\Delta r/c\right)}{2\pi\,\Delta r^3}e^{-i\omega\Delta r/c}$$

$$\Rightarrow\quad \boxed{\hat{p}\left(\underline{r}_A,\omega\right)=\frac{z_A}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{p}\left(x,y,0;\omega\right)\left(1+i\omega\frac{\Delta r}{c}\right)\frac{e^{-i\omega\Delta r/c}}{\Delta r^3}\,dxdy}$$

$$\Delta r=\sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+z_A^2}\ .$$

This is the *Rayleigh II integral*.

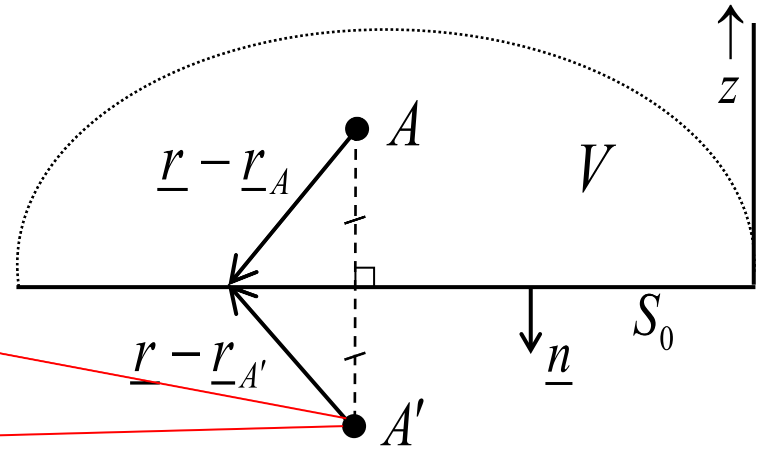
Rayleigh Integral



Rayleigh I Integral

An alternative choice for $\hat{\Gamma}$ in the Kirchhoff integral

$$\hat{p}\left(\underline{r}_A,\omega\right) = -\oint\!\!\!\oint_S \left[\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right)-\left(\hat{G}+\hat{\Gamma}\right)\nabla\hat{p}\right]\cdot\underline{n}\,dS$$



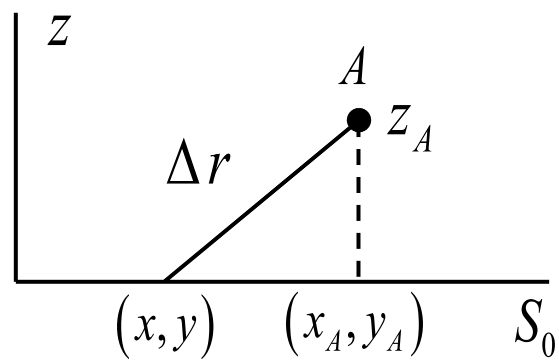
will cancel the term $\hat{p}\nabla\left(\hat{G}+\hat{\Gamma}\right)$.

By choosing:

$$\hat{G}(\underline{r}) = \frac{e^{-i\omega|\underline{r}-\underline{r}_A|/c}}{4\pi|\underline{r}-\underline{r}_A|} \quad \text{and} \quad \hat{\Gamma}(\underline{r}) = \frac{e^{-i\omega|\underline{r}-\underline{r}_{A'}|/c}}{4\pi|\underline{r}-\underline{r}_{A'}|}$$

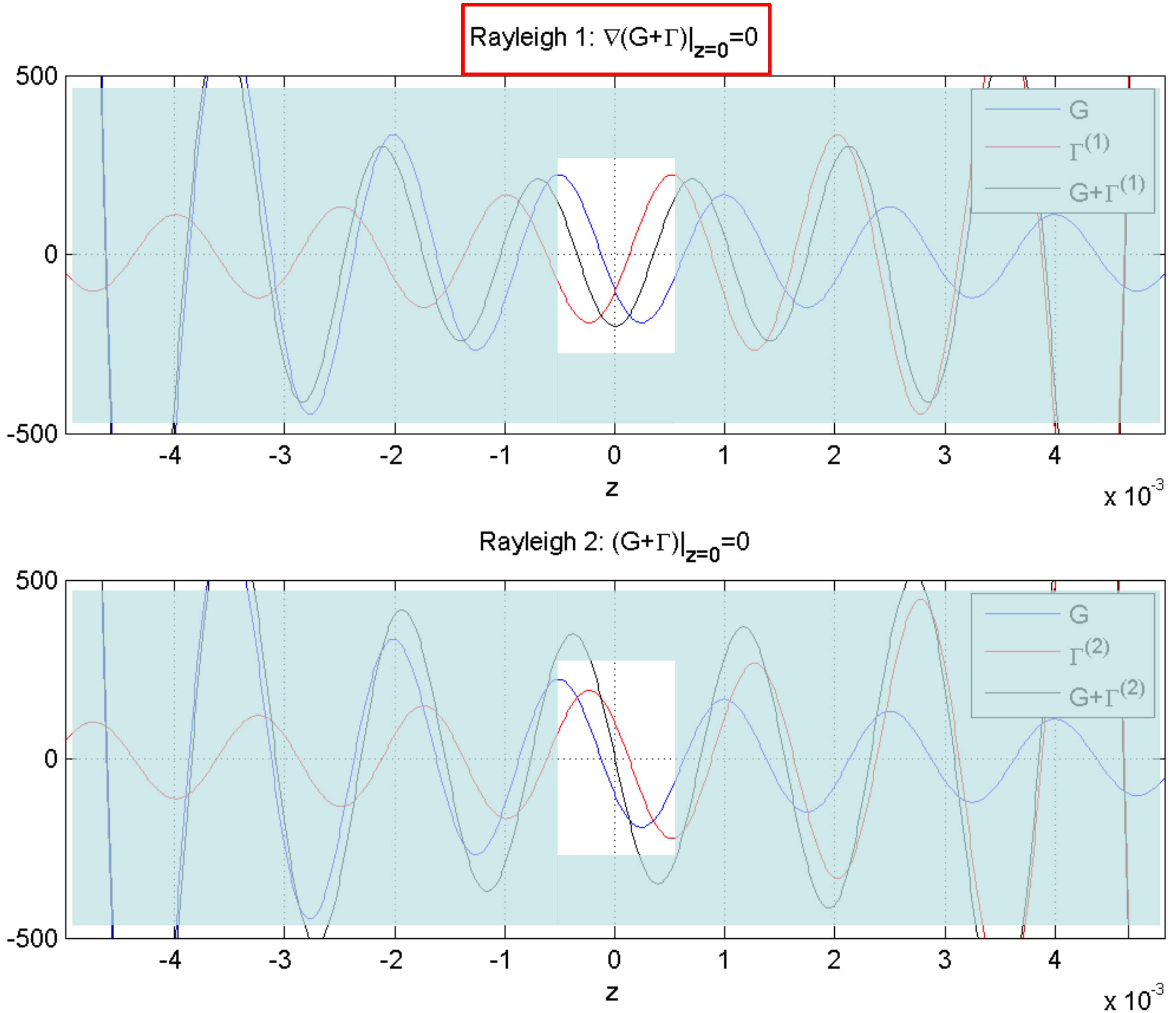
we obtain the *Rayleigh I integral*:

$$\boxed{\hat{p}\left(\underline{r}_A,\omega\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega\Delta r/c}}{\Delta r}\left(\frac{\partial\hat{p}}{\partial z}\right)_{z=0}dxdy}$$

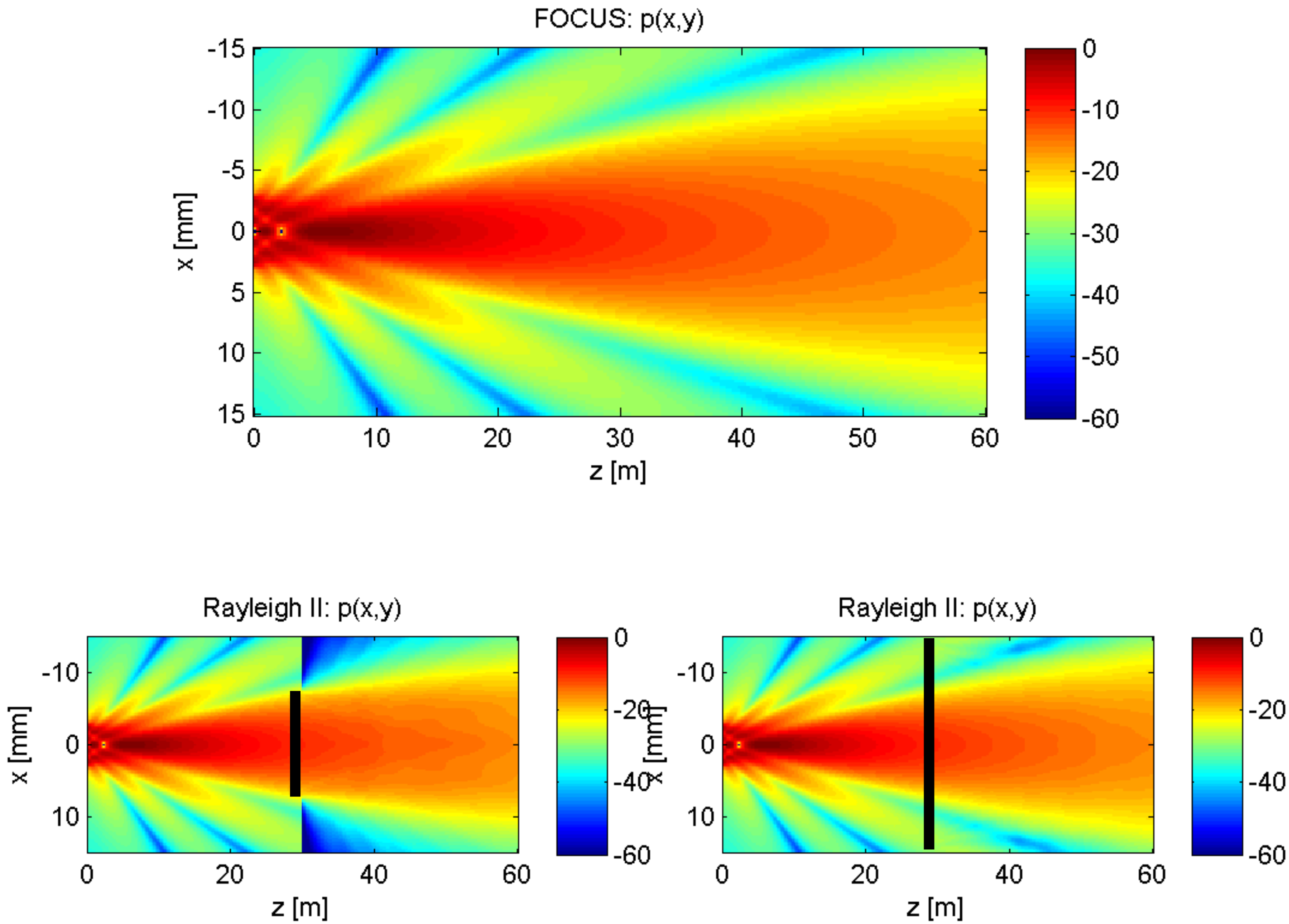


$$\Delta r = \sqrt{\left(x-x_A\right)^2+\left(y-y_A\right)^2+z_A^2}$$

Rayleigh Integral



Rayleigh II Example



Rayleigh II Integral as Spatial Convolution

The Rayleigh II integral:

$$\hat{p}(x_A, y_A, z_A; \omega) = \frac{\Delta z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(x, y, z; \omega) \, \frac{1 + i\omega(\Delta r/c)}{\Delta r^3} \, e^{-i\omega(\Delta r/c)} \, dxdy$$

with: $\Delta r = \sqrt{(x_A - x)^2 + (y_A - y)^2 + \Delta z^2}$, $\Delta z = z_A - z > 0$

can be written as:

$$\hat{p}(x_A, y_A, z_A; \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{Q}(x_A - x, y_A - y, \Delta z; \omega) \, \hat{p}(x, y, z; \omega) \, dxdy$$

with: $\hat{Q}(x, y, \Delta z; \omega) = \dfrac{\Delta z}{2\pi} \dfrac{\left(1 + \dfrac{i\omega}{c}\sqrt{x^2 + y^2 + \Delta z^2}\right)}{\left[x^2 + y^2 + \Delta z^2\right]^{3/2}} \, e^{-i(\omega/c)\sqrt{x^2 + y^2 + \Delta z^2}}$

$\hat{Q}(x, y, \Delta z; \omega)$ is the spatial convolution operator for forward extrapolation from the plane z to the plane $z_A = z + \Delta z$.

See also
George C. Sherman, "Application of the Convolution Theorem to Rayleigh's Integral Formulas," J. Opt. Soc. Am. 57, 546-547 (1967)

TUDelft

Helmholtz Equation in the (k_x, k_y)-domain

We define the double spatial Fourier transform of $\hat{p}(x, y, z; \omega)$:

$$\tilde{\hat{p}}(k_x, k_y; z; \omega) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)}\hat{p}(x, y, z; \omega)\, dxdy$$

So, double Fourier transformation of the Helmholtz equation to the (k_x, k_y)-domain:

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + \frac{\omega^2}{c^2}\hat{p} = 0 \quad \Rightarrow \quad \frac{\partial^2 \tilde{\hat{p}}}{\partial z^2} + \left(\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right)\tilde{\hat{p}} = 0$$

with $k_z \equiv \sqrt{(\omega/c)^2 - k_x^2 - k_y^2}$, we get: $\boxed{\dfrac{\partial^2 \tilde{\hat{p}}}{\partial z^2} + k_z^2 \tilde{\hat{p}} = 0}$.

Helmholtz Equation in the (k_x,k_y)-domain

Let us consider a wave field $\tilde{\tilde{p}}\left(k_x,k_y,z;\omega\right)$, generated by sources below the $z=z_0$ plane.

Then the source-free Helmholtz equation:

$$\frac{\partial^2 \tilde{\tilde{p}}}{\partial z^2}+k_z^2\tilde{\tilde{p}}=0 \qquad \text{with:} \qquad k_z=\sqrt{\frac{\omega^2}{c^2}-\left(k_x^2+k_y^2\right)}$$

is valid for all $z\geq z_0$.

Since waves are travelling in the positive z direction only, the above differential equation in z is readily solved for $z\geq z_0$, by:

$$\boxed{\tilde{\tilde{p}}\left(k_x,k_y,z_0+\Delta z;\omega\right)=\tilde{\tilde{p}}\left(k_x,k_y,z_0;\omega\right)e^{-ik_z\Delta z}} \quad , \qquad \Delta z>0$$

where $\tilde{\tilde{p}}\left(k_x,k_y,z_0\,;\omega\right)$ represents the double spatial Fourier transform of the observations made in the $z=z_0$ plane.

The factor $e^{-ik_z\Delta z}$ is the multiplicative forward extrapolation operator in the $\left(k_x,k_y,z\right)$-domain.

Evanescent Field

For $k_x^2 + k_y^2 > \dfrac{\omega^2}{c^2}$, we have

$$k_z = \sqrt{\frac{\omega^2}{c^2} - \left(k_x^2 + k_y^2\right)} = \pm i\sqrt{\left|\frac{\omega^2}{c^2} - \left(k_x^2 + k_y^2\right)\right|}$$

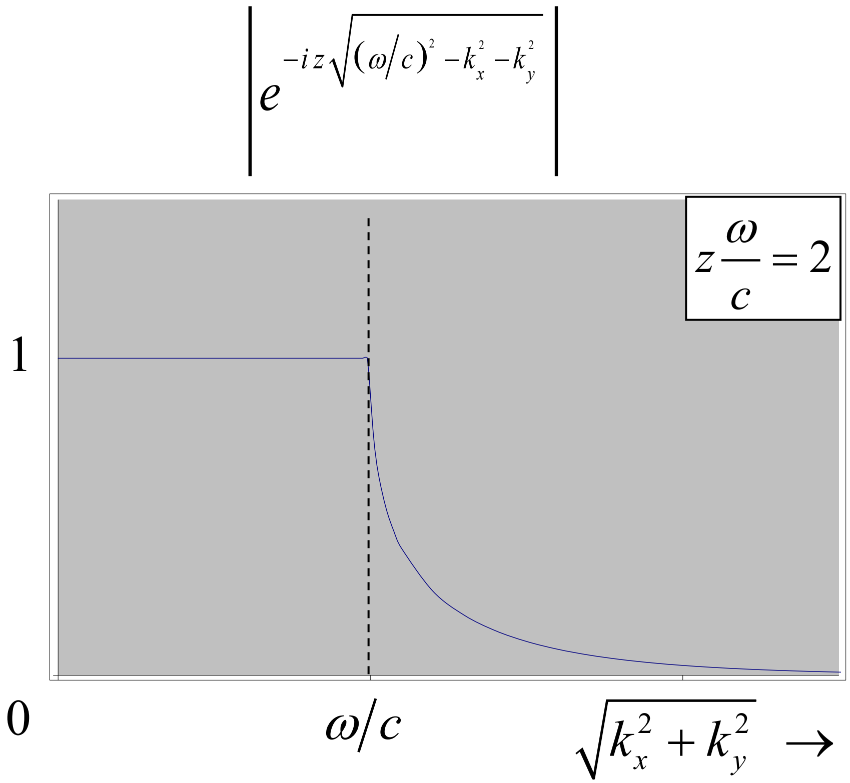
and: $$e^{-ik_z \Delta z} = e^{\pm \Delta z \sqrt{\left|(\omega/c)^2 - \left(k_x^2 + k_y^2\right)\right|}}$$

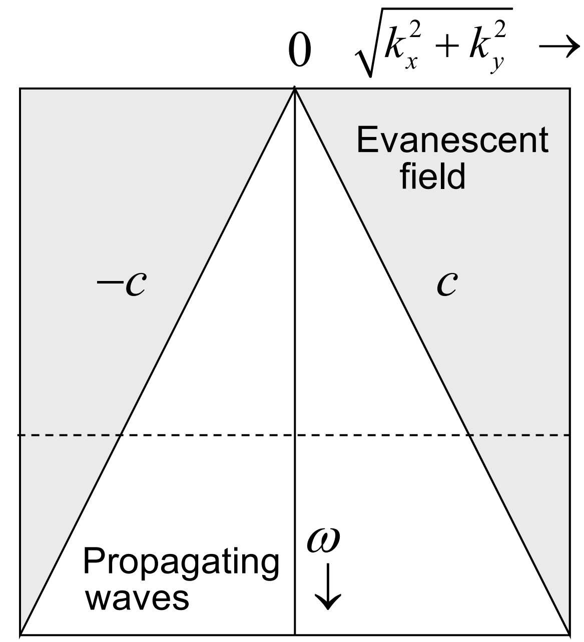
As the plus sign would be physically unacceptable in the half space $\Delta z \geq 0$, we get:

$$\tilde{\tilde{p}}\left(k_x, k_y, z\,;\omega\right) = \tilde{\tilde{p}}\left(k_x, k_y, z_0\,;\omega\right) e^{-\Delta z \sqrt{\left|(\omega/c)^2 - \left(k_x^2 + k_y^2\right)\right|}} \quad , \quad k_x^2 + k_y^2 > \frac{\omega^2}{c^2}$$

Any energy in $\tilde{\tilde{p}}\left(k_x, k_y, z_0\,;\omega\right)$ for which $\left(k_x^2 + k_y^2\right) > \omega^2 / c^2$, dies out very quickly with increasing Δz . This is called the *evanescent* field.

Evanescent Field





The evanescent field is observable only in 2-D plane-wave decompositions of wave-fields $\left(\text{remember } k_z \equiv \sqrt{(\omega/c)^2 - k_x^2 - k_y^2}\right)$.

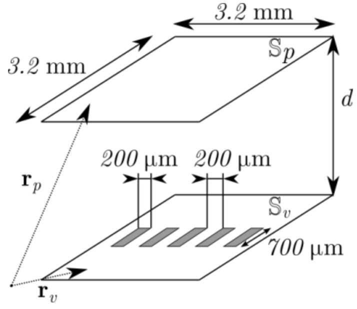
Example

This technique can be used to compute the velocity profile of an ultrasound transducer.

The pressure field at $z=a$ can be obtained from measurements at $z=0$ as follows:

$$\tilde{\tilde{p}}\left(k_x,k_y,z=a\,;\omega\right)=\tilde{\tilde{p}}\left(k_x,k_y,z=0\,;\omega\right)e^{-ik_z a} \quad ,$$

with $k_z = \sqrt{\frac{\omega^2}{c^2}-k_x^2-k_y^2}$, or $k_z=-i\sqrt{\left|\frac{\omega^2}{c^2}-k_x^2-k_y^2\right|}$ if $\frac{\omega^2}{c^2}<k_x^2+k_y^2$.

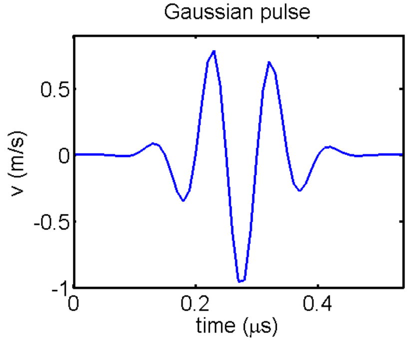


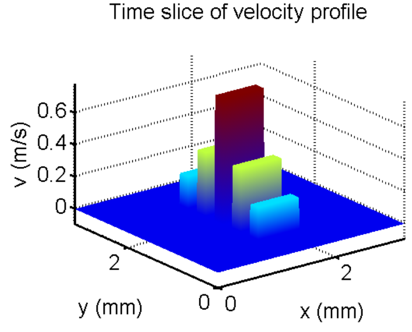
To obtain the field at $z=0$ from measurements at $z=a$,

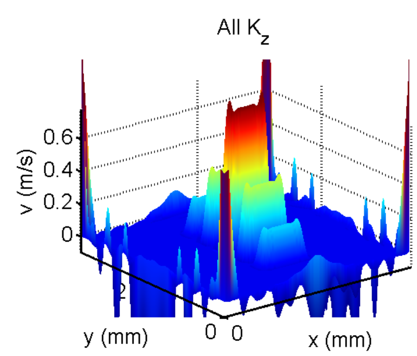
means we devide by $e^{-ik_z a}$, i.e.

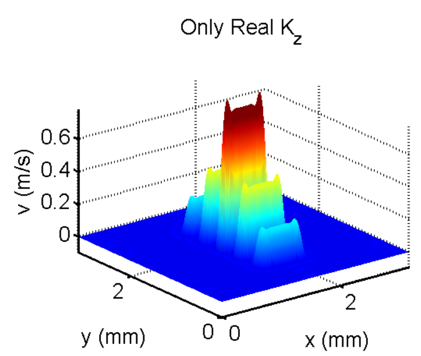
$$\tilde{\tilde{p}}\left(k_x,k_y,z=0\,;\omega\right)=\tilde{\tilde{p}}\left(k_x,k_y,z=a\,;\omega\right)e^{+ik_z a}.$$

However, problems may arise in case for $k_z=-i\sqrt{\left|\frac{\omega^2}{c^2}-k_x^2-k_y^2\right|}$ if $\frac{\omega^2}{c^2}<k_x^2+k_y^2$.









Rayleigh

- With Rayleigh, it is possible to efficiently model the wave field from one plane to the other.
- This propagation can be done forward and backward, as long as the medium is homogeneous, densely sampled, and stretches out to "infinity".
- However, in practice, "infinity" can not be met as well as that needle hydrophones have a wide but still limited openings angle.
- An alternative approach is to consider the Rayleigh 1 integral as an integral equation and measure the pressure field over a hemisphere.

Rayleigh

The Raleigh 1 can be rewritten as as spatial convolution of a Greens function with the unknown velocity field, i.e.

$$\hat{p}(\underline{r}_A,\omega) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega\frac{\Delta r}{c}}}{\Delta r}\left(\frac{\partial\hat{p}}{\partial z}\right)_{z=0} dxdy$$

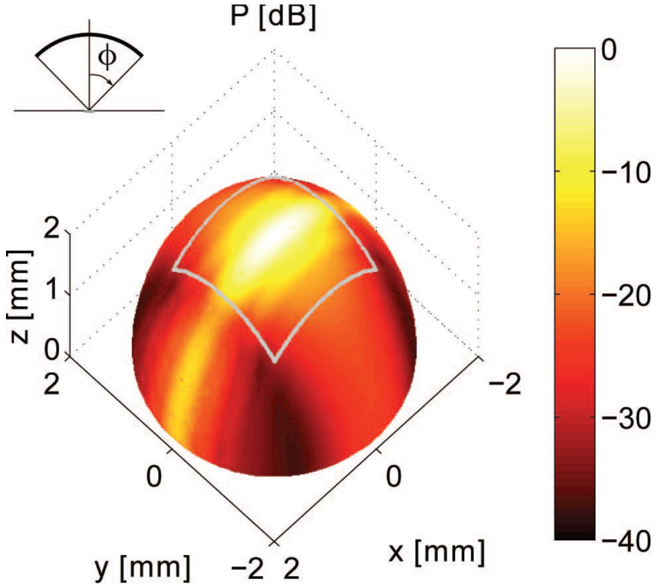
$$= 2i\omega\rho_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega\frac{\Delta r}{c}}}{\Delta r}v_z(x,y)\ dxdy$$

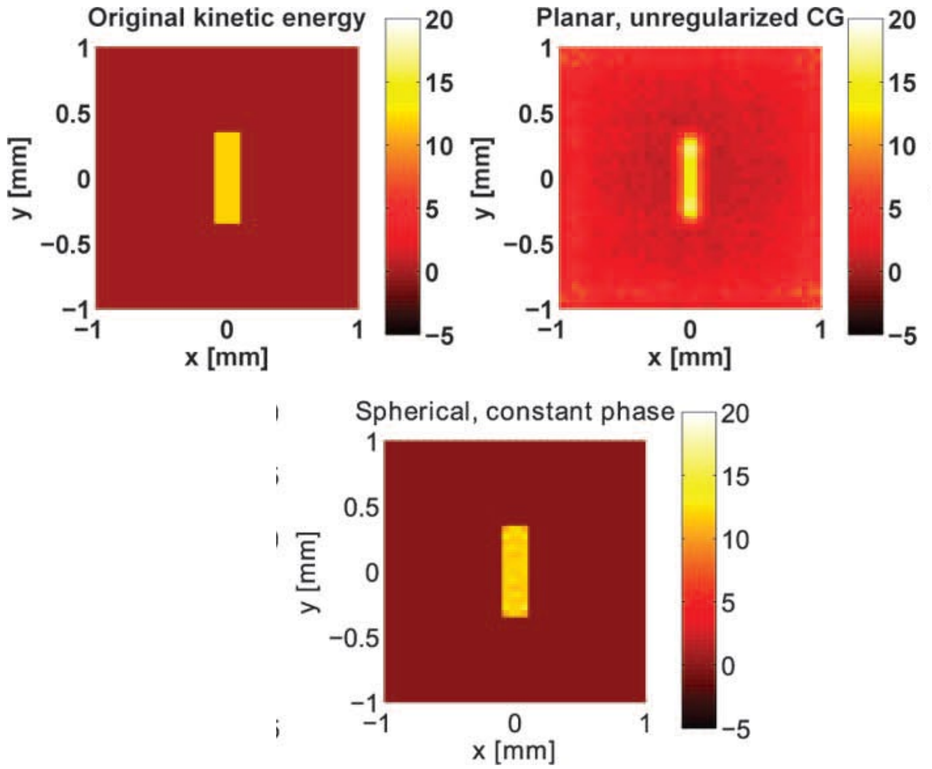
or as $\hat{p}(\underline{r}_A,\omega)=\mathbf{A}[v_z(x,y)]$, with known operator $\mathbf{A}$.

This linear problem can be solved via an iterative minimization scheme: $v_z^{(n+1)} = v_z^{(n)} + \alpha^{(n)}d^{(n)}$, even when the pressure field is measured on an arbitrary surface.

TUDelft

Example





TU Delft

Example

- Reconstruction of the velocity profile of a damaged IVUS transducer.